

Name: Solutions

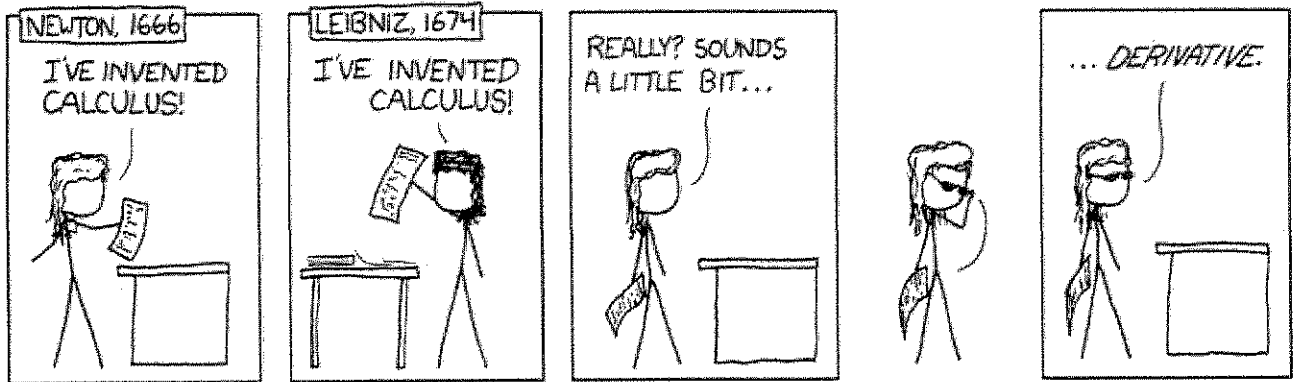
Signature: Solutions

Date: 3 June 2014

Do not start this exam until instructed; you will have 50 minutes to finish the exam. Show all your work to receive credit. No notes, books, calculators, phones or electronic devices are allowed on this exam.

There are 6 problems on this exam on 6 pages, in addition to this cover page.

Good luck!



From *xkcd*.

1. Compute the derivative of the function

$$f(x) = x + \frac{1}{x}$$

using the formal definition of the derivative; no credit will be given for any other method.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(x+h + \frac{1}{x+h}\right) - \left(x + \frac{1}{x}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \left[1 + \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right] \right] \\ &= 1 + \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x - (x+h)}{(x+h)x} \right] \\ &= 1 + \lim_{h \rightarrow 0} \frac{-1}{(x+h)x} \end{aligned}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

2. Find the derivatives of the following functions; do not simplify your answers.

(a)

$$f(x) = \frac{x^2 + 6}{x^2 + 6x + 6}$$

Quotient rule: $f'(x) = \frac{(x^2 + 6x + 6)(2x) - (x^2 + 6)(2x + 6)}{(x^2 + 6x + 6)^2}$

(b)

$$g(t) = \sqrt{t^2 + 1}$$

Chain rule: $g'(t) = \frac{d}{dt} (t^2 + 1)^{1/2}$
 $= \frac{1}{2} (t^2 + 1)^{-1/2} \cdot 2t$

(c)

$$h(w) = \sin(\cos(\sin w))$$

Chain rule
twice: $h'(w) = \cos(\cos(\sin w)) \cdot \frac{d}{dw} \cos(\sin w)$
 $= \cos(\cos(\sin w)) (-\sin(\sin w)) \cos w$

3. A ball is thrown from the top of a building, and its position above the ground is described by

$$s(t) = -16t^2 + 80t + 224$$

for $t \geq 0$ measured in seconds (s), and position in meters (m).

(a) When is the velocity zero?

(b) What are the velocity and acceleration of the ball when it hits the ground?

$$a) \quad v(t) = s'(t) = -32t + 80$$

$$v = 0 \Rightarrow -32t + 80 = 0 \Rightarrow t = \frac{80}{32} \text{ s}$$

$$t = \frac{5}{2} \text{ s}$$

$$b) \quad s(t) = 0 : \quad -t^2 + 5t + 14 = 0$$

$$t^2 - 5t - 14 = 0$$

$$(t - 7)(t + 2) = 0$$

$$t = 7, -2$$

$$v(7) = -32 \cdot 7 + 80 = -224 + 80 = -144$$

$$a = v' = -32$$

$$v = -144 \text{ m/s}$$

$$a = -32 \text{ m/s}^2$$

4. Find dy/dx if

$$\sin x + \cos y + y = y^2$$

$$\frac{d}{dx} \sin x + \frac{d}{dx} \cos y + \frac{d}{dx} y = \frac{d}{dx} y^2$$

$$\cos x - (\sin y) \frac{dy}{dx} + \frac{dy}{dx} = 2y \frac{dy}{dx}$$

$$\cos x = \frac{dy}{dx} (2y + \sin y - 1)$$

$$\frac{dy}{dx} = \frac{\cos x}{2y + \sin y - 1}$$

5. Find the linearization of the function

$$f(x) = \sqrt[4]{x}$$

at $x = 81$, and use it to estimate $\sqrt[4]{79}$.

$$\begin{aligned} L(x) &= f'(a)(x - a) + f(a) \\ &= f'(81)(x - 81) + f(81) \end{aligned}$$

$$\begin{aligned} f'(81) &= \left. \frac{1}{4} x^{-3/4} \right|_{x=81} \\ &= \frac{1}{4} (81)^{-3/4} \\ &= \frac{1}{4} \cdot 3^{-3} = \frac{1}{4 \cdot 27} = \frac{1}{108} \end{aligned}$$

$$L(x) = \frac{1}{108}(x - 81) + 3$$

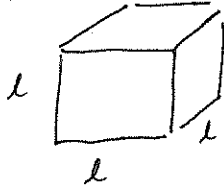
$$L(79) = 3 - \frac{2}{108} = 3 - \frac{1}{54}$$

$$\therefore \sqrt[4]{79} \approx 3 - \frac{1}{54}$$

Note $3 - \frac{1}{54} = 2.98148$

while $\sqrt[4]{79} \approx 2.9813$.

6. A cube of ice is melting. Suppose that the side length is decreasing at a rate of 1 in / min when the side length is 8 in. How quickly is the volume decreasing at this time? How quickly is the surface area decreasing at this time?



$$V = l^3$$

$$A = 6l^2$$

↳ 6 faces which are squares.

Know: $\frac{dl}{dt} = -1 \text{ in/min.}$

Want: $\frac{dV}{dt}$, $\frac{dA}{dt}$

$$\begin{aligned} \frac{dV}{dt} &= 3l^2 \cdot \frac{dl}{dt} \Rightarrow \frac{dV}{dt} = 3(8 \text{ in})^2 (-1 \text{ in/min}) \\ &= -192 \text{ in}^3/\text{min.} \end{aligned}$$

$$\frac{dA}{dt} = 12l \frac{dl}{dt} \Rightarrow \frac{dA}{dt} = -96 \text{ in}^2/\text{min.}$$

Volume: decreasing at a rate
of $192 \text{ in}^3/\text{min}$

Area: decreasing at a rate
of $96 \text{ in}^2/\text{min}$