# Name SOLUTIONS 

## PID

## Exam 1

## MTH 320, Friday February 15, 2019

Instructions: This exam is closed books, no calculators and no electronic devices of any kind. You are allowed one sheet of notes, front side only. There are four problems worth 25 points each. If a problem has multiple parts, it may be possible to solve a later part without solving the previous parts. Solutions should be written neatly and in a logically organized manner. Partial credit will be given if the student demonstrates an understanding of the problem and presents some steps leading to the solution. Correct answers with no work will be given no credit. The back sheets may be used as scratch paper but will not be graded for credit.

| 1 | 2 | 3 | 4 | Total |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

Problem 1. ( 25 pts ) For each statement, circle $T$ if it is true and $F$ if it is false. If true, give a brief explanation (a complete proof is not required), and if false, give a counterexample.
a. $(T / F)$ If $B \subset \mathbb{R}$ and $t$ is an upper bound for $B$ then $\sup (B \cup\{t\})=t$.

True. Since $t$ is an upper bound for $B$ and $t \leq t$ it follows that $t$ is an upper bound for $B \cup\{t\}$. If $s$ is an upper bound for $B \cup\{t\}$, then since $t \in B \cup\{t\}$, it follows that $t \leq s$. Therefore, by definition of least upper bound $t=\sup (B \cup\{t\})$.
b. ( $T / F$ ) If $\sum a_{n}$ converges absolutely and $\left(b_{n}\right)$ converges then $\sum a_{n} b_{n}$ converges absolutely.

True. Since $\left(b_{n}\right)$ converges this implies that the sequence is bounded. Let $M>0$ be such that $\left|b_{n}\right|<M$ for all $n \in \mathbb{N}$. We want to compare $\left|a_{n} b_{n}\right|$ with $M\left|a_{n}\right|$. It follows that

$$
0 \leq\left|a_{n} b_{n}\right|=\left|a_{n}\right|\left|b_{n}\right|<M\left|a_{n}\right| .
$$

Since $\sum\left|a_{n}\right|$ converges, the limit laws imply that $\sum M\left|a_{n}\right|$ converges to $M\left(\sum a_{n}\right)$. Therefore, by the Comparison Test, $\sum\left|a_{n} b_{n}\right|$ converges and thus, $\sum a_{n} b_{n}$ converges absolutely.
c. $(T / F)$ If the sequence $\left(a_{n}\right)$ converges and the sequence $\left(a_{n} b_{n}\right)$ converges then the sequence $\left(b_{n}\right)$ converges.
False. If $a_{n}=0$ for all $n \in \mathbb{N}$ is the constant sequence of all 0 then it converges to 0 . Let $b_{n}=n$ and notice that $b_{n}$ diverges since $b_{n}$ is unbounded. However, $a_{n} b_{n}=0$ for all $n \in \mathbb{N}$ which converges to 0 .
d. $(T / F)$ If $A$ is non-empty and bounded above then $\sup A \in A$.

False. Let $A=(0,1)$. We know that $\sup A=1$ however $1 \notin A$.
e. ( $T / F$ ) If $0 \leq a_{n} \leq 7$ for all $n \geq 10$, then $\left(a_{n}\right)$ has a convergent subsequence.

True. Let $M=\max \left(\left|a_{1}\right|,\left|a_{2}\right|, \ldots,\left|a_{9}\right|, 8\right)$. It follows that $\left(a_{n}\right)$ is bounded by $M$, that is, $\left|a_{n}\right|<M$ for all $n \in \mathbb{N}$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\left(a_{n}\right)$.

Problem 2. Consider the sequence $\left(\frac{1}{n^{2}+5}\right)$.
a. (15 pts) Show directly that $\left(x_{n}\right)$ converges to 0 . (Use only the definition of convergence and the Archimedean Property.)
Let $\epsilon>0$ be given.
We consider two cases. If $\epsilon>\frac{1}{5}$ then for any $n \in \mathbb{N}$

$$
n^{2}+5>5>0 \quad \Longrightarrow \quad 0<\frac{1}{n^{2}+5}<\frac{1}{5}<\epsilon
$$

If $0<\epsilon \leq 1 / 5$ then, by the Archimedean Property, there exists $N \in \mathbb{N}$ such that

$$
\sqrt{\frac{1}{\epsilon}-5}<N
$$

If $n>N$ then

$$
\begin{array}{r}
\sqrt{\frac{1}{\epsilon}-5}<n \\
\Longrightarrow \quad 0<\frac{1}{n^{2}+5}<n^{2}+5 \\
\Longrightarrow \quad
\end{array}
$$

In both cases, we have found that if $n>N$ then $\left|\frac{1}{n^{2}+5}-0\right|<\epsilon$. Therefore, $\left(\frac{1}{n^{2}+5}\right)$ converges to 0 .
b. (10 pts) Does that series $\sum_{n=1}^{\infty} \frac{1}{n^{2}+5}$ converge? Justify your answer. (You may use theorems proved in class.)
In class we proved that the series $\sum \frac{1}{n^{2}}$ converges. Moreover, for all $n \in \mathbb{N}$ we have that

$$
n^{2}+5>n^{2}>0 \quad \Longrightarrow \quad 0<\frac{1}{n^{2}+5}<\frac{1}{n^{2}}
$$

Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^{2}+5}$ converges.

Problem 3. Let $a \in \mathbb{R}$ and define the set $S=\{x \in \mathbb{Q}: x<a\} \subset \mathbb{R}$.
a. ( 5 pts ) Prove that $S$ has no lower bound.

Suppose that $S$ is bounded below. Let $q \in \mathbb{R}$ be such that $q \leq x$ for all $x \in S$ which moreover implies that $q<a$. By the Archimedean Property there exists $M \in \mathbb{N}$ such that $-q<M$. Since $-M \in \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$ it follows that $-M \in \mathbb{Q}$. Further, $-M<q<a$ which implies that $-M \in S$ is an element less than the lower bound. Thus, we have reached a contradiction.
b. (10 pts) Prove that $\sup S=a$.

By definition of $S, a$ is an upper bound for $S$.
Let $\epsilon>0$ be given. Then, $a-\epsilon<a$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $a-\epsilon<q<a$ which further implies that $q \in S$. Therefore, $a$ is the least upper bound.
c. (10 pts) Find a sequence $\left(a_{n}\right)$ such that $a_{n} \in S$ for all $n \in \mathbb{N}$ and $\left(a_{n}\right)$ converges to $a$.

Notice that $a-\frac{1}{n}<a$ for all $n \in \mathbb{N}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for all $n \in \mathbb{N}$ there exists $q_{n} \in \mathbb{Q}$ such that

$$
a-\frac{1}{n}<q_{n}<a .
$$

Thus, $q_{n} \in S$ for all $n \in \mathbb{N}$. We know that $\lim _{n \rightarrow \infty} a-\frac{1}{n}=a=\lim _{n \rightarrow \infty} a$. Therefore by the Squeeze Theorem, $\left(q_{n}\right)$ converge to $a$.

Problem 4. Define a sequence recursively by $x_{1}=2$ and

$$
x_{n+1}=\frac{1}{2}\left(x_{n}+\frac{3}{x_{n}}\right) \quad \text { for } \quad n \in \mathbb{N} .
$$

a. (10 pts) Prove that $\left(x_{n}\right)$ is bounded below by $\sqrt{3}$ for all $n \in \mathbb{N}$.

We prove this by induction. For the base case $n=1$, we have that $x_{1}=2=\sqrt{4}>\sqrt{3}$.
Suppose that $x_{n}>\sqrt{3}$. It follows that

$$
\begin{array}{rlr}
0 & <x_{n}-\sqrt{3} \\
\Longrightarrow & 0 & <\left(x_{n}-\sqrt{3}\right)^{2}=x_{n}^{2}-2 x_{n} \sqrt{3}+3 \\
\Longrightarrow & 2 x_{n} \sqrt{3} & <x_{n}^{2}+3 \\
\Longrightarrow \quad \sqrt{3} & <\frac{1}{2}\left(x_{n}+\frac{3}{x_{n}}\right)=x_{n+1} & \\
& &
\end{array}
$$

Therefore by induction, $x_{n}>\sqrt{3}$ for all $n \in \mathbb{N}$.
b. (10 pts) Show that the sequence is monotone decreasing.

From a. we have that for all $n \in \mathbb{N}$

$$
\begin{aligned}
& \sqrt{3}<x_{n} \\
& \Longrightarrow \quad 3<x_{n}^{2} \quad \text { (square both sides) } \\
& \Longrightarrow \quad \frac{3}{x_{n}}<x_{n} \quad \quad \text { (divide by } x_{n}>0 \text { ) } \\
& \Longrightarrow \quad x_{n}+\frac{3}{x_{n}}<x_{n}+x_{n}=2 x_{n} \quad\left(\text { add } x_{n}\right) \\
& \Longrightarrow \quad \frac{1}{2}\left(x_{n}+\frac{3}{x_{n}}\right)<x_{n} \\
& \therefore \quad x_{n+1}<x_{n} \text {. } \\
& \text { (divide by } 2 \text { ) }
\end{aligned}
$$

Therefore, $x_{n}$ is monotone decreasing.
c. (5 pts) By the Monotone Convergence Theorem the $\left(x_{n}\right)$ converges.

Compute the limit: $x=\lim _{n \rightarrow \infty} x_{n}$.
Since $\left(x_{n}\right)$ converges to $x$ then the subsequence $\left(x_{n+1}\right)$ also converges to $x$. Moreover since $\sqrt{3}<x_{n}$ for all $n \in \mathbb{N}$, the Order Limit Law implies that $\sqrt{3} \leq x$. Therefore, we can apply the Algebraic Limit Law to the recursive relation:

$$
\begin{aligned}
\lim x_{n+1}= & \lim \frac{1}{2}\left(x_{n}+\frac{3}{x_{n}}\right) & \\
x=\frac{1}{2}\left(x+\frac{3}{x}\right) & & \text { (we use that } \left.\lim x_{n}=x \neq 0\right) \\
& \Longrightarrow \quad x^{2}-3=0 \quad & \Longrightarrow \quad x= \pm \sqrt{3} .
\end{aligned}
$$

Since we already know that $x \geq \sqrt{3}$ we conclude that $x=\sqrt{3}$.

