Asymptotic Results for the Havriliak-Negami Dielectric Model

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Abstract

The properties of the Havriliak-Negami (H-N) dielectric model, given by

\[ \varepsilon(x, t) = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + i \omega \tau x / v} \]

and \( r \) is the ratio of the permittivities \( r = \varepsilon_0 / \varepsilon_\infty \), are in terms of the speed, we have \( \sqrt{\tau} = \varepsilon_\infty / \varepsilon_0 \). If \( f \) represents the incident pulse at \( x = 0 \), and \( F(x) = \mathcal{F} [f(t)] \), the electric field can be expressed analytically in the H-N medium as

\[ E(x, t) = \frac{1}{2 \varepsilon_\infty} \int_{-\infty}^{\infty} F(s) e^{-i\omega(x - s^2/2v)} ds, \quad \varepsilon > \tau \sqrt{2v} \]  

(1)

We choose \( \varepsilon > \tau \sqrt{2v} \) so that no additional singularities can be introduced by the phase argument for \( x > \tau \sqrt{2v} \). To obtain asymptotic solutions at shallow and large depths, we look at the leading order behavior of the phase argument in each case.

Short Depth Response

The transient behavior of the pulse occurs in the skin-depth region, \( x = \mathcal{O}(\varepsilon_0 \tau) \). The asymptotic behavior of \( E(x, t) \) near the wavefront is determined. The leading order solution will be valid for \( x \ll \varepsilon_0 \tau \). By a simple scaling argument, it can be shown that the limit corresponds to \( x \gg 1 \) in terms of the Laplace variable \( s \), and so shallower depth solutions will be determined by contributions from the integrand in the limit \( s \to \infty \). Upon expanding the phase argument of (1) we have

\[ s \left( 1 - \frac{x^2}{2\varepsilon_0 \tau} \right) \approx s \left( \frac{x^2}{2\varepsilon_0 \tau} - A(s) \right) \approx O(s^{-1/2}) \]

with \( A(s) = 1 + \frac{1}{x^2/2\varepsilon_0 \tau} \). To obtain the asymptotic impulse response \( F \), we set \( F(s) = 1 \). Simplifying the Bromwich contour onto the branch cut, we use contributions from \( s \approx \sqrt{\varepsilon_0 \tau} \) to write this as a real integral

\[ \Phi(x, t) = \frac{1}{\sqrt{\varepsilon_0 \tau}} \int_{-\infty}^{\infty} e^{i\omega(x - s^2/2v)} e^{s^2/2v} \cos \left( \omega(t - s) \right) ds \]

(2)

This impulse response can be convolved with a given incident pulse \( f(t) \) to produce the approximate electric field at shallow depths. For illustration, we use a square wave \( f(t) = 1 \) for \( |t| / H < t < |t| / H \), with \( H \) the width of the pulse and \( \tau \) is the center frequency, defined in terms of the inverse Laplace transform

\[ \chi(t) = \frac{1}{H} \left( e^{-|t|/H} - e^{-|t|/H} \right) \]

(3)

where \( \chi(t) \) is the impulse response. If we choose \( A(s) = 1 + i \omega \tau / \sqrt{\varepsilon_0 \tau} \), the approximate electric field obeys the subcharacteristic wave equation \( \partial \psi / \partial x = c \psi \), with \( c = 1 + \frac{1}{x^2/2\varepsilon_0 \tau} \). This leads to an alternate expansion of the phase.

Large Depth Response

Past the skin-depth (\( x \gg \varepsilon_0 \tau \)) we estimate the late time behavior of \( E(x, t) \) near the subcharacteristic ray \( x = x_0 \) by assuming that \( x \gg \tau \), or equivalently that \( x \gg 1 \). This leads to an alternate expansion of the phase.

Determining Parameters

In [3] it was shown that for a symmetric pulse (\( \alpha = 1 \)), the long-depth electric field satisfies an advection-diffusion equation with speed given by the zero-frequency permittivity \( c_0 = 1 / \tau \sqrt{\varepsilon_0 / \varepsilon_\infty} \). For asymmetric pulses (\( \alpha < 1 \)), we find the mean electric field lies on the subcharacteristic ray \( x = x_0 / \alpha \), when the pulse is symmetric, the peak electric field will coincide with this mean value. We then define the time at which the mean electric field occurs using the mean value theorem,

\[ x_0 = \frac{\int_{-\infty}^{\infty} F(s) \chi(s) ds}{\int_{-\infty}^{\infty} \chi(s) ds} \]

(4)

Upon constructing this quantity, we find that \( x_0 \) is a linear function of \( x \), with slope \( 1/\alpha \). The intercept can be computed by setting \( x = 0 \), and then \( x = c_0 \). We may then also be expressed in terms of the incident pulse

\[ F(x) = f(0) \left( 1 + \frac{x}{x_0} \right) \]

If we can similarly determine the parameters \( c_0 \) and \( \tau \), the late asymptotic response can be used to determine H-N parameters (\( \alpha, \beta \)). In figure (5), the numerical solution of the electric field with \( \alpha = 0.75 \) represents a data set in the long-time limit and is plotted on a log-log scale. The first plot (upper left panel) shows the fit when the correct parameters are chosen, and we see good agreement. When the correct value of \( \alpha \) is found, but \( \beta \) is not correct, the curves will be parallel for \( x \rightarrow \infty \), or in top right panel. In both lower panels, the value of \( \alpha \) is incorrect; however, we cannot determine much, whether \( \beta \) is correct or not in this case. Thus, we can only proceed with determining \( \beta \) after finding \( \alpha \).

Conclusions

We have constructed asymptotic approximations for the shallow and large-depth behaviors of the electric field in dielectric media that are described by the Havriliak-Negami dielectric model. We have also demonstrated how these results can be used to determine \( c_0 \) and \( \tau \) for a given set of TDR data. Determining the \( c_0 \) and \( \tau \) remains an open question.

References


