Topological Properties of Activity Orders for Matroids

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1. Matroids and the External Order

Let M be a matroid on a finite set E and let

$$\mathcal{B}(M) = \{B \subseteq E \mid B \text{ is a base of } M\}.$$

Ex. 1. The *vector matroid* $M = (V, +, \cdot,)$ has $E = V$, a finite dimensional vector space over a finite field, and

$$\mathcal{B}(V) = \{B| \ B \text{ is a basis of } V\}.$$

2. The *graph matroid* $M = G$ where G is a connected graph has $E = E(G)$ (edges of G) and

$$\mathcal{B}(G) = \{E(T) \mid T \text{ is a spanning tree of } G\}.$$

Here we will make no distinction between $F \subseteq E$ and the spanning graph of G with edge set F.

The *independent sets of M* are

$$\mathcal{I}(M) = \{I \mid I \subseteq B \text{ for some base } B\}.$$

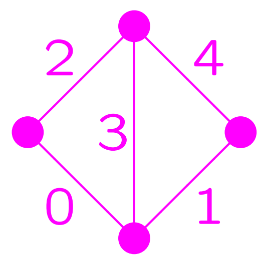
The *circuits of M* are

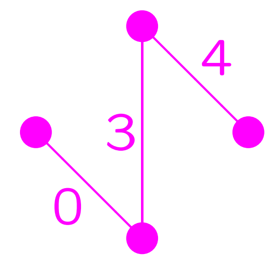
$$\mathcal{C}(M) = \{C \mid C \text{ a minimal dependent set}\}.$$

If $M = G$ then a circuit of M is just a cycle in G.

From now on all matroids will be *ordered*, i.e., there will be a total order on E. Then $F \subseteq E$ has *external active set*

$$\mathrm{Act}(F) = \{e \in E \mid e = \min C \text{ where } C \in \mathcal{C}(F \cup e)\}.$$

Ex. Let $G =$ 

and $F =$ 

Then $\mathrm{Act}(F) = \{1\}$.

The *external lattice of the ordered matroid M* is

$$L(M) = \mathcal{B}(M) \uplus \hat{0}$$

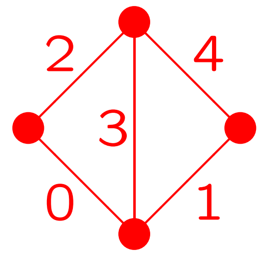
where $\hat{0}$ is a unique minimal element and for bases $A, B \in \mathcal{B}(M)$

$$A \leq B \quad \text{iff} \quad A \subseteq B \cup \mathrm{Act}(B).$$

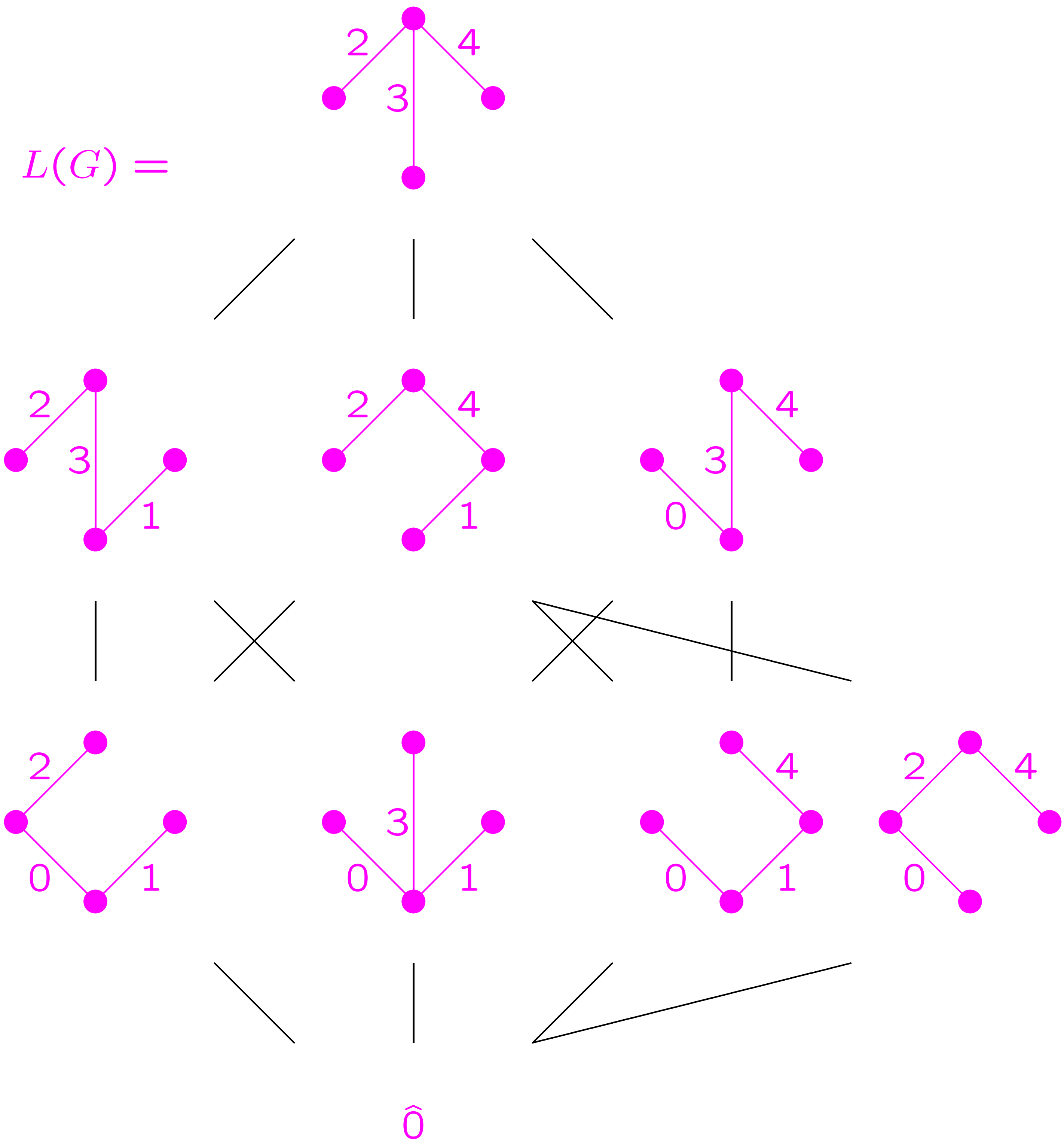
Theorem 1 (Las Vergnas) *$L(M)$ is a lattice with unique maximal element T which is the lexicographically largest base (writing sets in increasing order). $L(M)$ is also ranked with rank function*

$$\rho(B) = |\,\mathrm{Act}(B)| + 1.$$

$G =$



$L(G) =$



2. The Order Complex

An *abstract simplicial complex* is a finite family Δ of finite nonempty sets called *simplices* such that

$$S \in \Delta \text{ and } R \subseteq S \ (R \neq \emptyset) \text{ implies } R \in \Delta.$$

Let P be any finite poset with a unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. The *order complex of P* is

$$\Delta(P) = \{C \mid C \text{ is a chain in } P \setminus \hat{0}, \hat{1}\}.$$

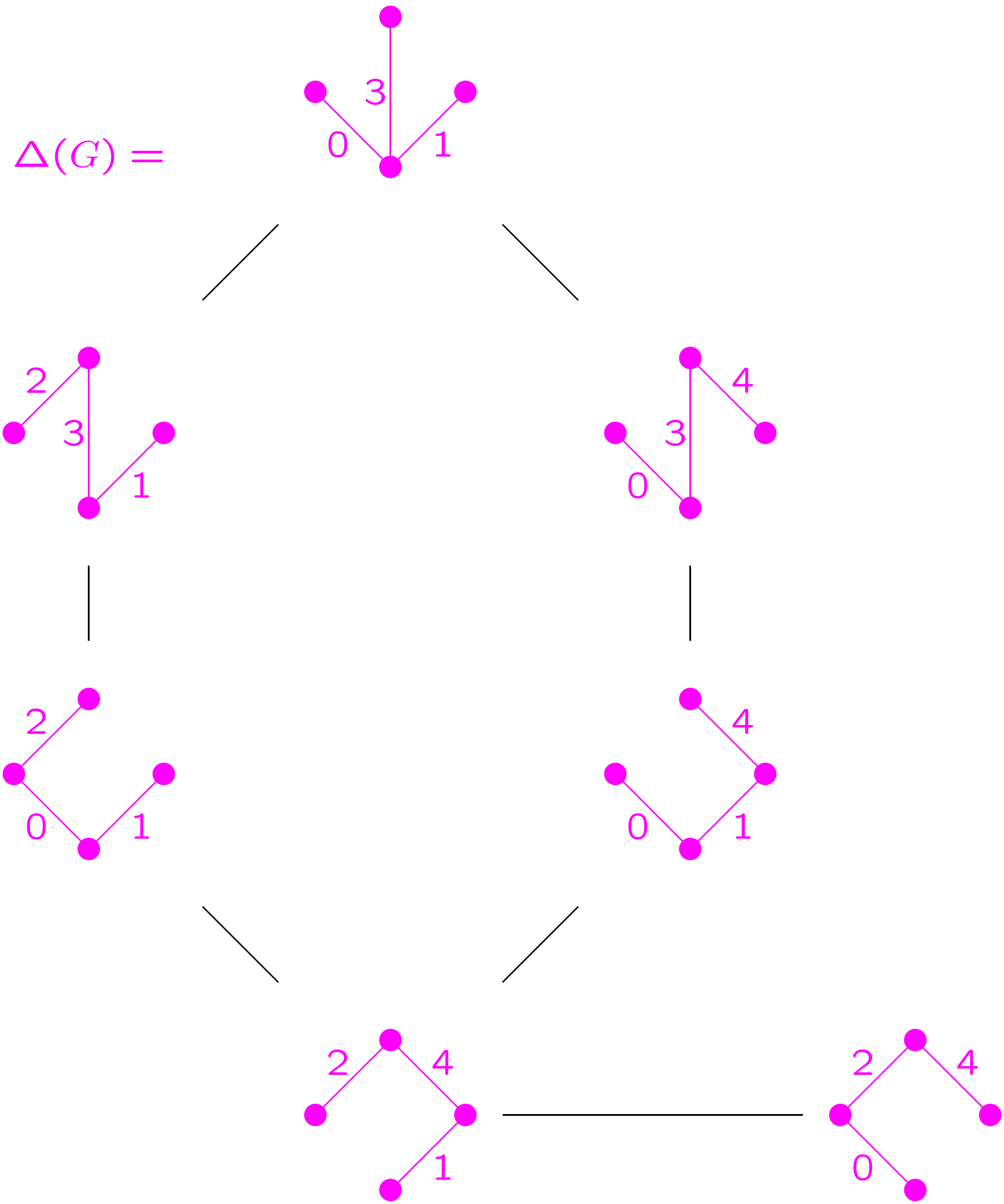
Note that this is an abstract simplicial complex since a subset of a chain is again a chain.

We have the correspondences

P	$\Delta(P)$
1-chain	vertex
2-chain	edge
3-chain	triangle
k-chain	simplex of dimension $k-1$

If M is an ordered matroid then $\Delta(M) = \Delta(L(M))$.

$\Delta(G) =$



3. The Main Theorem

If M is a matroid on E then the *dual of M* is the matroid M^* defined by

$$\mathcal{B}(M^*) = \{B^* \mid B^* = E - B \text{ for some } B \in \mathcal{B}(M)\}.$$

Ex. For our example graph $E(G) = \{0, 1, 2, 3, 4\}$ so

$$\begin{aligned} \mathcal{B}(M) &= \{012, 013, 014, 024, 123, 124, 034, 234\} \\ \mathcal{B}(M^*) &= \{34, 24, 23, 13, 04, 03, 12, 01\} \end{aligned}$$

If M is a matroid on E and $F \subseteq E$ then the *restriction of M to F* is the matroid $M|F$ defined by

$$\mathcal{B}(M|F) = \{B \cap F \mid B \in \mathcal{B}(M) \text{ and } |B \cap F| \text{ maximum}\}.$$

Ex. (cont) Let $F = T = 234 = \hat{1}$ in $L(G)$

$$\mathcal{B}(M^*|T) = \{34, 24, 23\}.$$

If M is a matroid then the *matroid complex* $[M]$ is the abstract simplicial complex whose simplices are the nonempty independent sets of M.

Ex. (cont) We have a homotopy equivalence

$$[M^*|T] = \{34, 24, 23, 2, 3, 4\} \simeq S^1 \text{ (a 1-sphere)}$$

Theorem 2 (B & S) *If $T = \hat{1}$ in $L(M)$ then*

$$\Delta(M) \simeq [M^*|T].$$

4. Applications

Denote the Möbius function of $L(M)$ by

$$\mu(M) := \mu_{L(M)}(\hat{0}, T).$$

So $\mu(M)$ is the reduced Euler characteristic of $\Delta(M)$

$$\mu(M) = \sum_{i \geq -1} (-1)^i \dim \tilde{H}_i(\Delta(M)).$$

The *rank of M* is $\mathrm{rk}(M) := |B|$ for any $B \in \mathcal{B}(M)$.

Corollary 3 (B & S) *Let* $\mathrm{rk}(M) = r > 1$.

(i) If there is $B \in \mathcal{B}(M)$ with $B \subseteq E - T$, then

$$\tilde{H}_i(M) = \{0\} \text{ for all } i \geq 0 \text{ and } \mu(M) = 0.$$

(ii) If $E - S$ contains a base of M for all $S \subset T$ but not for $S = T$, then

$$\tilde{H}_i(M) = \begin{cases} \mathbb{Z} & \text{if } i = r-2, \\ \{0\} & \text{else,} \end{cases} \quad \text{and } \mu(M) = (-1)^{r-2}.$$

Proof (i) By definition $E - B \in \mathcal{B}(M^*)$ and we're given $E - B \supseteq T$. So $\Delta(M) \simeq [M^*|T] \simeq (r-1)$-ball and we are done.

(ii) Similarly $\Delta(M) \simeq [M^*|T] \simeq (r-2)$-sphere. ■

A *star* is a complete bipartite graph of the form $K_{1,n}$.

Corollary 4 (B & S) *Consider a matroid obtained by ordering the edges of the complete graph K_r, $r > 2$.*

(i) If T is not a star then then $\mu(K_r) = 0$.

(ii) If T is a star then $\mu(K_r) = (-1)^{r-3}$.

Proof (i) If T is not a star, then $K_r - E(T)$ is connected. So $K_r - E(T)$ contains a spanning tree and thus a base for K_r. Now use the previous corollary. (ii) is similar. ■

Note that $\Delta(M)$ is not necessarily shellable: Take an ordering of $E(K_r)$ so that the $r-1$ largest edges all meet at a vertex v. So every edge of $E - T$ is active and

$$\dim \Delta(K_r) = \rho(L(K_r)) - 2 = |\,\mathrm{Act}(T)| - 1 = \binom{r-1}{2} - 1.$$

But from Corollary 4, $\Delta(K_r)$ has non-zero homology in dimension $r - 3$.

Theorem 5 (Björner) *The complex $[M]$ is shellable for any matroid M.*

5. An Open Problem

Las Vergnas also defined an *internal order* $\leq_I$ on M in a way equivalent to defining

$$A \leq_I B \text{ iff } E - B \leq E - A \text{ in } L(M^*).$$

If $L_I(M)$ is the resulting lattice, then it is isompor-phic to the poset dual of $L(M^*)$ and so has the same homology and Möbius function.

One can combine the external and internal orders as follows. Define the *internal-external order* $\leq_{IO}$ on $\mathcal{B}(M)$ by

$$A \leq_{IO} B \text{ iff } A \leq B \text{ or } A \leq_I B.$$

It follows from results of Las Vergnas that this is a well-defined partial order.

Question 6 *What can be said about the homology and Möbius function of the corresponding poset $L_{IO}(M)$?*