Symmetric Functions in Noncommuting Variables

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0. History

1936	M. C. Wolf
1969	G M. Bergman & P. M. Cohn
1972	P. Doubilet
1981	M.-P. Schützenberger & A. Lascoux
1995	I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, & J.-I. Thibon
1998	S. Fomin & C. Greene
2000 & 2001	D. Gebhard & BES
2001 & 2002	MHR
2003	MHR & BES
2003	MHR & C. Reutenhauer

1. Symmetric Functions (SFs) Defined

Let $\mathbb{Q}[[\mathbf{x}]]$ and $\mathbb{Q}\langle\langle\mathbf{x}\rangle\rangle$ be the algebras of formal power series in *commuting* and *noncommuting* variables $\mathbf{x} = \{x_1, x_2, \ldots\}$, respectively. Any $g \in \mathfrak{S}_m$ (symmetric group) acts on f in either algebra by

$$gf(x_1, x_2, \ldots) = f(x_{g1}, x_{g2}, \ldots)$$

where $gi = i$ for $i > m$. Then f is *symmetric* if it is invariant under all g in all $\mathfrak{S}_m$. Let $\Lambda(\mathbf{x})$ and $\Pi(\mathbf{x})$ be the subalgebras of $\mathbb{Q}[[\mathbf{x}]]$ and $\mathbb{Q}\langle\langle\mathbf{x}\rangle\rangle$, respectively, of symmetric power series of bounded degree.

Bases for $\Lambda[\mathbf{x}]$ are indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. The *monomial SFs* are

$$m_\lambda = \sum(\text{monomials of exponent } \lambda).$$

Define *power sum* and *elementary SFs* by

$$\begin{aligned} p_n &= x_1^n + x_2^n + \cdots \\ e_n &= \sum(\text{square-free monomials of degree } n) \end{aligned}$$

and multiplicativity.

Ex. For $\lambda = (3, 3, 1)$ we have

$$\begin{aligned} m_{(3,3,1)} &= x_1^3 x_2^3 x_3 + x_1^3 x_2 x_3^3 + x_1 x_2^3 x_3^3 + \cdots \\ p_{(3,3,1)} &= (x_1^3 + x_2^3 + \cdots)^2 (x_1 + x_2 + \cdots) \\ e_{(3,3,1)} &= (x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots)^2 (x_1 + x_2 + \cdots) \end{aligned}$$

Bases for $\Pi[\mathbf{x}]$ are indexed by set partitions $\pi = B_1/B_2/\ldots/B_l \vdash [n]$ where $[n] = \{1,\ldots,n\}$. Define

$$\begin{aligned} m_\pi &= \sum x_{i_1} x_{i_2} \cdots x_{i_n} \text{ where} \\ &\quad i_j = i_k \text{ iff } j,k \text{ are in the same block of } \pi. \\ p_\pi &= \sum x_{i_1} x_{i_2} \cdots x_{i_n} \text{ where} \\ &\quad i_j = i_k \text{ if } j,k \text{ are in the same block of } \pi. \\ e_\pi &= \sum x_{i_1} x_{i_2} \cdots x_{i_n} \text{ where} \\ &\quad i_j \neq i_k \text{ if } j,k \text{ are in the same block of } \pi. \end{aligned}$$

Ex. For $\pi = 13/2$ we have

$$\begin{aligned} m_{13/2} &= x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + \cdots, \\ p_{13/2} &= x_1x_2x_1 + x_2x_1x_2 + \cdots + x_1^3 + x_2^3 + \cdots \\ e_{13/2} &= x_1x_1x_2 + x_1x_2x_2 + x_1x_3x_2 + \cdots \end{aligned}$$

For $\lambda = (\lambda_1,\ldots,\lambda_l) = (1^{c_1}, 2^{c_2}, \ldots, n^{c_n})$ let

$$\lambda! = \lambda_1!\lambda_2!\cdots\lambda_l! \quad \text{and} \quad \lambda^! = c_1!c_2!\cdots c_n!$$

The *type* of set partition π is the integer partition

$$\lambda(\pi) = (|B_1|, |B_2|, \ldots, |B_l|).$$

Theorem 1 *Let $\rho : \mathbb{Q}\langle\langle x\rangle\rangle \to \mathbb{Q}[[x]]$ be the forgetful map and $\lambda(\pi) = \lambda$. Then*

$$\begin{aligned} \rho(m_\pi) &= \lambda^! m_\lambda, \\ \rho(p_\pi) &= p_\lambda, \\ \rho(e_\pi) &= \lambda! e_\lambda. \end{aligned}$$

In the commuting case, the *complete homogeneous SFs* are defined by multiplicativity and

$$h_n = \sum_{\lambda \vdash n} m_\lambda .$$

Ex. For $\lambda = (3,3,1)$ we have

$$h_{(3,3,1)} = (x_1^3 + x_1^2 x_2 + x_1 x_2 x_3 + \cdots)^2 (x_1 + x_2 + \cdots).$$

For noncommuting variables, define

$$h_\pi = \sum_L m_\sigma$$

where the sum is over all linear orderings L of the elements in each block of $\sigma \wedge \pi$.

Ex. For $\pi = 13/2$ we have

$$h_{13/2} = m_{1/2/3} + m_{12/3} + 2m_{13/2} + m_{1/23} + 2m_{123}.$$

Theorem 2 *Applying the forgetful map gives*

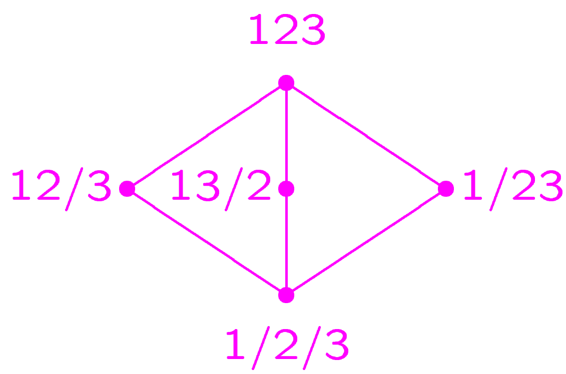
$$\rho(h_\pi) = \pi! h_{\lambda(\pi)}.$$

2. Basis Change

Let Π_n be the lattice of all $\pi \vdash [n]$ ordered by *refinement*, i.e., if $\pi = B_1/\dots/B_k$ and $\sigma = C_1/\dots/C_l$ then $\pi \leq \sigma$ if each B_i is contained in some C_j.

Ex. We have

$\Pi_3 =$



Let $\pi \wedge \sigma$ be the *greatest lower bound* or *meet* of π and σ. Also let $\hat{0} = 1/2/\dots/n$ and μ denote the minimum and Möbius function of Π_n, respectively.

Theorem 3 (Doubilet) *We have*

$$(1)\ p_\pi = \sum_{\sigma \geq \pi} m_\sigma,$$

$$(2)\ m_\pi = \sum_{\sigma \geq \pi} \mu(\pi,\sigma) p_\sigma,$$

$$(3)\ e_\pi = \sum_{\sigma \wedge \pi = \hat{0}} m_\sigma,$$

$$(4)\ m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi,\sigma)}{\mu(\hat{0},\sigma)} \sum_{\tau \leq \sigma} \mu(\tau,\sigma) e_\tau.$$

Proof (1) & (3) follow directly from the definitions. (2) & (4) now follow by Möbius inversion. ■

3. MacMahon Symmetric Functions

Given variable sets $\dot{\mathbf{x}} = \{\dot{x}_1, \dot{x}_2, \ldots\}$, $\ddot{\mathbf{x}} = \{\ddot{x}_1, \ddot{x}_2, \ldots\}$, ..., $\mathbf{x^{(n)}} = \{x_1^{(n)}, x_2^{(n)}, \ldots\}$, we let $g \in S_m$ act on a $f(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}) \in \mathbb{Q}[[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}]]$ *diagonally*:

$$gf(\dot{x}_1, \ddot{x}_1, \ldots, \dot{x}_2, \ddot{x}_2, \ldots) = f(\dot{x}_{g1}, \ddot{x}_{g1}, \ldots, \dot{x}_{g2}, \ddot{x}_{g2}, \ldots).$$

The monomial

$$\dot{x}_1^{a_1} \ddot{x}_1^{b_1} \cdots x_1^{(n)c_1} \dot{x}_2^{a_2} \ddot{x}_2^{b_2} \cdots x_2^{(n)c_2} \cdots$$

has *multiexponent*

$$\begin{aligned} \vec{\lambda} &= \{\lambda^1, \lambda^2, \ldots\} \\ &:= \{[a_1, b_1, \ldots, c_1], [a_2, b_2, \ldots, c_2], \ldots\} \end{aligned}$$

as well as *multidegree*

$$\begin{aligned} \vec{m} &= [m_1, m_2, \ldots, m_n] \\ &:= [a_1, b_1, \ldots, c_1] + [a_2, b_2, \ldots, c_2] + \cdots \end{aligned}$$

and we write $\vec{\lambda} \vdash \vec{m}$.

Ex. For $\dot{x}_1^2 \ddot{x}_1 \dot{x}_2^3$ we have

$\vec{\lambda} = \{[2,1],[3,0]\}$ and $\vec{m} = [2,1] + [3,0] = [5,1]$.

The *algebra of MacMahon symmetric functions*, $\mathcal{M}$, is the set of all elements in $\mathbb{Q}[[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}]]$ which are symmetric and of bounded multidegree. Bases for $\mathcal{M}$ are indexed by vector partitions $\vec{\lambda}$.

Define the *monomial MacMahon SF* by

$m_{\vec{\lambda}} =$ sum of all monomials with multiexponent $\vec{\lambda}$.

Define *power sum* & *elementary MacMahon SFs* by

$$\begin{aligned} p_{[a,b,\dots,c]} &= \dot{x}_1^a \ddot{x}_1^b \cdots x_1^{(n)c} + \dot{x}_2^a \ddot{x}_2^b \cdots x_2^{(n)c} \cdots \\ e_{[a,b,\dots,c]} &= \sum(\text{multidegree } [a,b,\dots,c] \text{ dottings} \\ &\qquad \text{of squarefree monomials}), \\ h_{[a,b,\dots,c]} &= \sum(\text{multidegree } [a,b,\dots,c] \text{ dottings} \\ &\qquad \text{of all monomials}), \end{aligned}$$

and multiplicativity. (In h, repetitions of a variable are considered distinct, leading to multiplicities.)

Ex. For $\vec{\lambda} = \{[2,1],[3,0]\}$ we have

$$\begin{aligned} m_{\{[2,1],[3,0]\}} &= \dot{x}_1^2 \ddot{x}_1 \dot{x}_2^3 + \dot{x}_1^3 \dot{x}_2^2 \ddot{x}_2 + \cdots \\ p_{\{[2,1],[3,0]\}} &= (\dot{x}_1^2 \ddot{x}_1 + \dot{x}_2^2 \ddot{x}_2 + \cdots)(\dot{x}_1^3 + \dot{x}_2^3 + \cdots) \\ e_{\{[2,1],[3,0]\}} &= (\dot{x}_1 \dot{x}_2 \ddot{x}_3 + \dot{x}_1 \ddot{x}_2 \dot{x}_3 + \ddot{x}_1 \dot{x}_2 \dot{x}_3 + \cdots) \\ &\qquad (\dot{x}_1 \dot{x}_2 \dot{x}_3 + \dot{x}_1 \dot{x}_2 \dot{x}_4 + \cdots) \\ h_{\{[2,1],[3,0]\}} &= (3\dot{x}_1^2 \ddot{x}_1 + \dot{x}_1^2 \ddot{x}_2 + 2\dot{x}_1 \ddot{x}_1 \dot{x}_2 + \cdots) \\ &\qquad (\dot{x}_1^3 + \dot{x}_1^2 \dot{x}_2 + \dot{x}_1 \dot{x} 2 \dot{x}_3 + \cdots) \end{aligned}$$

Let $\langle \cdot \rangle$ denote vector space span and consider

$$\mathcal{M}_{[1^n]} = \langle m_{\vec{\lambda}} \ : \ \text{all } \vec{\lambda} \vdash [1^n] \rangle.$$

Ex. For $n = 3$ we have

$$\mathcal{M}_{[1,1,1]} = \langle m_{\{[1,1,1]\}}, m_{\{[1,1,0],[0,0,1]\}}, m_{\{[1,0,1],[0,1,0]\}}, m_{\{[0,1,1],[1,0,0]\}}, m_{\{[1,0,0],[0,1,0],[0,0,1]\}} \rangle.$$

Theorem 4 (Rosas) *The map*

$$\Phi : \bigoplus_{n \geq 0} \mathcal{M}_{[1^n]} \to \Pi(\mathbf{x})$$

given by linearly extending

$$\dot{x}_i \ddot{x}_j \cdots x_k^{(n)} \overset{\Phi}{\mapsto} x_i x_j \cdots x_k$$

is an isomorphism of vector spaces. In fact

$$b_{\{\lambda^1, \lambda^2, \ldots, \lambda^l\}} \overset{\Phi}{\mapsto} b_{B_1/B_2/\ldots/B_l}$$

where $b = m$, p, e or h and λ^i is the characteristic vector of B_i.

Ex. $m_{\{[1,0,1],[0,1,0]\}} \overset{\Phi}{\mapsto} m_{13/2}$ since

$$\dot{x}_1 \,\dddot{x}_1\, \ddot{x}_2 + \dot{x}_2 \,\dddot{x}_2\, \ddot{x}_1 + \dot{x}_1 \,\dddot{x}_1\, \ddot{x}_3 + \cdots$$

$$\overset{\Phi}{\mapsto} x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + \cdots$$

4. Schur Symmetric Functions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ has a *shape*, also denoted λ, consisting of l left-justified rows with λ_i dots in row i.

Ex. Partition $\lambda = (3, 3, 2, 1)$ has shape

$$\lambda = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}.$$

A *(semistandard) Young tableau T of shape λ*, written $\lambda(T) = \lambda$, is obtained by replacing each dot of the shape of λ with a positive integer so that rows weakly increase and columns strictly increase.

Ex. A Young tableau of shape $(3, 3, 2, 1)$ is

$$T = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 3 & 6 & \\ 5 & & \end{array}.$$

The *Schur function*, s_λ, is

$$s_\lambda = \sum_{\lambda(T)=\lambda} M_T \quad \text{where} \quad M_T = \prod_{i \in T} x_i.$$

Ex. If $\lambda = (2,1)$ then

$$T \quad : \quad \begin{array}{ll} 1 & 1 \\ 2 & \end{array} \;,\; \begin{array}{ll} 1 & 2 \\ 2 & \end{array} \;,\; \ldots \;,\; \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \;,\; \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \;,\; \ldots$$

$$s_{(2,1)} \;=\; x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_3 + \cdots$$

If $\lambda, \vec{m} \vdash d$ then a *dotted Young tableau $\dot{T}$ of shape λ and multidegree $\vec{m}$* is obtained from T by putting one dot on m_1 elements of T, two dots on m_2 elements of T, etc. The corresponding *MacMahon Schur function* is

$$S_\lambda^{\vec{m}} = \sum_{\lambda(\dot{T})=\lambda} M_{\dot{T}} \quad \text{where} \quad M_{\dot{T}} = \prod_{i^{(j)} \in T} x_i^{(j)}.$$

Ex. If $\lambda = (2,1)$ and $\vec{m} = [1,1,1]$ then

$$\dot{T} \quad : \quad \begin{array}{ll} \dot{1} & \ddot{1} \\ \dddot{2} & \end{array} \;,\; \begin{array}{ll} \ddot{1} & \dot{1} \\ \dddot{2} & \end{array} \;,\; \ldots \;,\; \begin{array}{ll} \dddot{1} & \ddot{1} \\ \dot{2} & \end{array} \;,\; \ldots$$

$$S_{(2,1)}^{[1,1,1]} \;=\; \dot{x}_1 \ddot{x}_1 \dddot{x}_2 + \dot{x}_1 \ddot{x}_1 \dddot{x}_2 + \cdots + \ddot{x}_1 \dddot{x}_1 \dot{x}_2 + \cdots$$

The MacMahon Schur functions have many of the same properties as do the regular Schur functions, such as a Robinson-Schensted-Knuth correspondence.

Theorem 5 (Jacobi-Trudi Determinant) *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we have*

$$s_\lambda = |h_{\lambda_i - i + j}|.$$

The analogue of this theorem for MacMahon symmetric functions is as follows.

Theorem 6 (R-S) *Given a partition λ and vector $\vec{m}$ with $\lambda, \vec{m} \vdash n$, we have*

$$S_\lambda^{\vec{m}} = \mathcal{T}^{\vec{m}} \left| \sum_{\vec{q} \vdash \lambda_i - i + j} h_{\vec{q}} \right|$$

where $\mathcal{T}^{\vec{m}}$ extracts the terms of multidegree $\vec{m}$ from the determinant.

Note that when $\vec{m}$ has only one component, then this reduces to the ordinary Jacobi-Trudi Theorem. And when $\vec{m} = [1^n]$ then this gives a noncommuting variable analogue.

Ex. Consider $\lambda = (2,1)$.

If $\vec{m} = [3]$, then

$$s^{[3]}_{(2,1)} = \mathcal{T}^{[3]} \begin{vmatrix} \sum_{\vec{q} \vdash 2} h_{\vec{q}} & \sum_{\vec{q} \vdash 3} h_{\vec{q}} \\ \sum_{\vec{q} \vdash 0} h_{\vec{q}} & \sum_{\vec{q} \vdash 1} h_{\vec{q}} \end{vmatrix}$$

$$= h_{[2]} h_{[1]} - h_{[3]} h_{[0]}$$

$$= h_{(2,1)} - h_3$$

as normal.

Now if $\vec{m} = [1,1,1]$, then

$$s^{[1,1,1]}_{(2,1)} = \mathcal{T}^{[1,1,1]} \begin{vmatrix} \sum_{\vec{q} \vdash 2} h_{\vec{q}} & \sum_{\vec{q} \vdash 3} h_{\vec{q}} \\ \sum_{\vec{q} \vdash 0} h_{\vec{q}} & \sum_{\vec{q} \vdash 1} h_{\vec{q}} \end{vmatrix}$$

$$= \mathcal{T}^{[1,1,1]} \big\{ (h_{[1,1,0]} + h_{[1,0,1]} + h_{[0,1,1]}) \cdot (h_{[1,0,0]} + h_{[0,1,0]} + h_{[0,0,1]}) - h_{[1,1,1]} h_{[0,0,0]} \big\}$$

$$= h_{[1,1,0]} h_{[0,0,1]} + h_{[1,0,1]} h_{[0,1,0]} + h_{[0,1,1]} h_{[1,0,0]} - h_{[1,1,1]}.$$

5. Open Problems

A. What is the analogue of the quotient of alternants definition of a Schur function for the $S_\lambda^{[1^n]}$?

B. We have only defined $S_\lambda^{[1^n]}$ for λ an *integer* partition. Is there some way to define noncommuting variable functions $S_\pi^{[1^n]}$ for all *set* partitions π?

C. What does all this have to do with representation theory?

D. Can one make further progress on the $(\mathbf{3+1})$-free conjecture of Stanley and Stembridge using symmetric functions in noncommuting variables?