On a rank-unimodality conjecture of Morier-Genoud and Ovsienko

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A *partially ordered set* or *poset* is a finite set P together with a binary relation $\trianglelefteq$ such that for all $x, y, z \in P$:

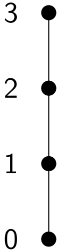
1. (reflexivity) $x \trianglelefteq x$,
2. (antisymmetry) $x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$,
3. (transitivity) $x \trianglelefteq y$ and $y \trianglelefteq z$ implies $x \trianglelefteq z$.

We also adopt the usual conventions for inequalities. For example, $x \triangleleft y$ means $x \trianglelefteq y$ and $x \neq y$.

If $x, y \in P$ then *x is covered by y* or *y covers x* if $x \triangleleft y$ and there is no z with $x \triangleleft z \triangleleft y$. The *Hasse diagram* of P is the (directed) graph with vertices P and an edge from x up to y if x is covered by y.

Example: The Chain.

The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ and $\trianglelefteq$ is the usual $\leq$ on the integers.



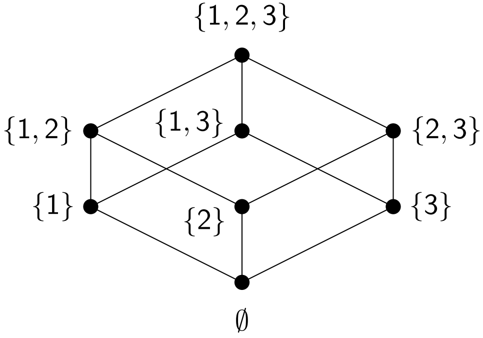
C_3

Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \trianglelefteq T$ if and only if $S \subseteq T$.

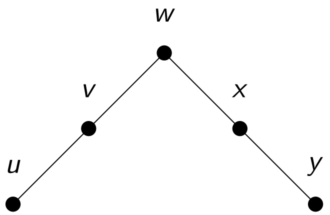


B_3

Example: Lambda.

The poset lambda has elements $\Lambda = \{u, v, w, x, y\}$ with covers

$$u \lhd v \lhd w \rhd x \rhd y.$$



Λ

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y \lhd x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y \rhd x$.
Ex. Λ has minimal elements u, y and maximal element w.

A poset *has a zero* if it has a unique minimal element, $\hat{0}$.
Ex. C_n has $\hat{0} = 0$ and B_n has $\hat{0} = \emptyset$.

If $x \unlhd y$ in P then an *x–y chain of length n* is

$$C : x = x_0 \lhd x_1 \lhd \ldots \lhd x_n = y.$$

Say C is *saturated* if all the $\lhd$ are covers.
Ex. In B_n we have

$$C : \emptyset \subset \{1, 3\} \subset \{1, 2, 3\}$$

is an $\emptyset$–$\{1, 2, 3\}$ chain of length 2. It is not saturated. The chain

$$C' : \emptyset \subset \{3\} \subset \{1, 3\} \subset \{1, 2, 3\}$$

is saturated of length 3.

A poset P is *ranked* if it has a $\hat{0}$ and, for any $x \in P$, the lengths of all saturated $\hat{0}$–x chains have the same length. The common length of these chains is the *rank of x* and denoted $\mathrm{rk}\, x$.

Ex. Poset C_n is ranked and $\mathrm{rk}\, k = k$.

Poset B_n is ranked and $\mathrm{rk}\, S = \#S$.

Poset Λ is not ranked since it does not have a $\hat{0}$.

If P is ranked then the *rank numbers of P* are

$$r_k(P) = \text{number of elements of } P \text{ at rank } k.$$

Ex. Poset C_n has $r_k(C_n) = 1$.

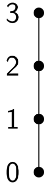
Poset B_n has $r_k(B_n) = \binom{n}{k}$.

If P is ranked and q is a variable then P has *rank generating function*

$$r(q; P) = \sum_k r_k(P) q^k.$$

Ex. Poset C_n has $r(C_n; q) = 1 + q + q^2 + \cdots + q^n$.

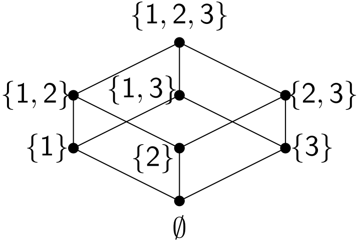
Poset B_n has $r(q; B_n) = (1 + q)^n$.



C_3

rank	$r_k(C_3)$
3	1
2	1
1	1
0	1

$$r(q; C_3) = 1 + q + q^2 + q^3$$



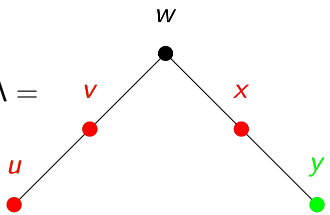
B_3

rank	$r_k(B_3)$
3	1
2	3
1	3
0	1

$$r(q; B_3) = 1 + 3q + 3q^2 + q^3 = (1+q)^3$$

A *(lower order) ideal* of P is a subset $I \subseteq P$ with the property that

$$y \in I \quad \text{and} \quad x \trianglelefteq y \implies x \in I.$$

$\Lambda =$ 

$\{u, v, y\}$ is an ideal

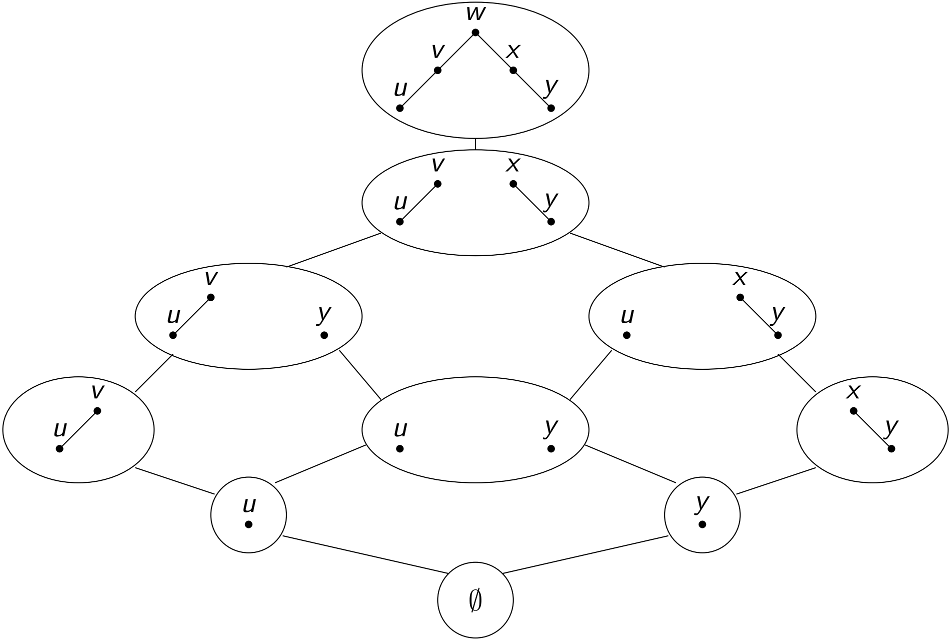
$\{u, v, x\}$ is not an ideal

The *lattice of ideals* of P is the set

$$L(P) = \{I \subseteq P \mid I \text{ is an ideal of } P\}$$

partially ordered by inclusion. The lattice $L(P)$ is ranked and

$$\mathrm{rk}\, I = \# I.$$



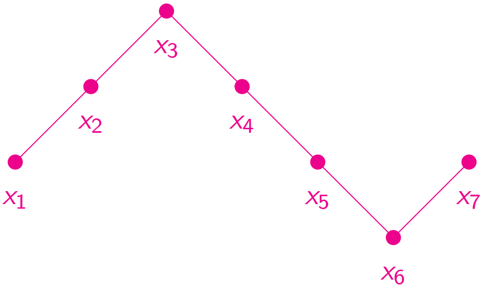
$L(\Lambda)$

Let $\alpha = (a, b, \ldots)$ be a *composition*, that is, a sequence of positive integers called *parts*. A *fence* is a poset $F = F(\alpha)$ with elements $x_1, \ldots, x_n$ and covers

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots \ .$$

Ex.

$F(2,3,1) =$



Note that $\Lambda = F(2,2)$.

The maximal chains of F are called *segments*. Note that if $\alpha = (\alpha_1, \alpha_2, \ldots)$ then

$$n = \#F(\alpha) = 1 + \sum_i \alpha_i.$$

Let $L = L(\alpha)$ be the lattice of order ideals of $F(\alpha)$. These lattices can be used to compute mutations in a cluster algebra on a surface with marked points.

Who	When	What
Propp	2005	perfect mathings on snake graphs
Yurikusa	2019	perfect matchings of angles
Schiffler	2008, 2010	T-paths
Schiffler and Thomas	2009	T-paths
Propp	2005	lattice paths on snake graphs
Claussen	2020	lattice paths of angles
Claussen	2020	S-paths

Lattice $L(\alpha)$ is ranked with rank function $\mathrm{rk}\, I = \# I$. We let

$$r_k(\alpha) = \#\{I \in L(\alpha) \mid \mathrm{rk}\, I = k\}.$$

We will also use the rank generating function

$$r(q;\alpha) = \sum_k r_k(\alpha) q^k.$$

This generating function was used by Morier-Genoud and Ovsienko to define q-analogues of rational numbers. Call a sequence $a_0, a_1, \ldots$ or its generating function *unimodal* if there is an index m with

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots .$$

Conjecture (Morier-Genoud and Ovsienko, 2020)

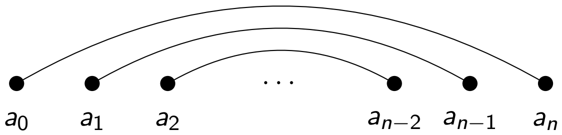
For any α we have that $r(q;\alpha)$ is unimodal.

Previous work: Gansner (1982), Munarini and Salvi (2002), Claussen (2020).

Call sequence $a_0, a_1, \ldots, a_n$ *symmetric* if, for all $k \le n/2$,

$$a_k = a_{n-k}.$$

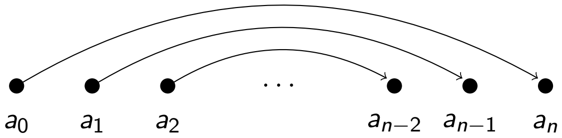
Using an edge between a and b if $a = b$, symmetry looks like



Call the sequence *top heavy* or *bottom heavy* if, for all $k \le n/2$,

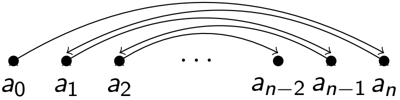
$$a_k \le a_{n-k} \quad \text{or} \quad a_k \ge a_{n-k}, \qquad \text{respectively.}$$

Using an arc from a to b if $a \le b$, top heavy looks like



Call the sequence *top interlacing (TI)* if

$$a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq a_2 \leq \ldots \leq a_{\lceil n/2 \rceil}.$$



Call the sequence *bottom interlacing (BI)* if

$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq a_{n-2} \leq \ldots \leq a_{\lfloor n/2 \rfloor}.$$

Note that interlacing implies unimodality and heaviness.

Conjecture (MSS)

Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$.

(a) *If s is even, then $r(q;\alpha)$ is BI.*

(b) *Suppose $s \geq 3$ is odd.*

- (i) *If $\alpha_1 > \alpha_s$ then $r(q;\alpha)$ is BI.*
- (ii) *If $\alpha_1 < \alpha_s$ then $r(q;\alpha)$ is TI.*
- (iii) *If $\alpha_1 = \alpha_s$ then let $\alpha' = (\alpha_2, \ldots, \alpha_{s-1})$. If $r(q;\alpha')$ is symmetric, BI or TI then $r(q;\alpha)$ is symmetric, TI, or BI, respectively.*

Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ *and suppose that for some* t *we have*

$$\alpha_t > \sum_{i \neq t} \alpha_i.$$

Then $r(q;\alpha)$ *is unimodal.*

Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ *where for some* t

$$\alpha_t = 1 + \sum_{i \neq t} \alpha_i.$$

If $r(q;\alpha)$ *is rank symmetric, BI, or TI then so is* $r(q;\beta)$ *where*

$$\beta = (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + a, \alpha_{t+1}, \ldots, \alpha_s)$$

for any $a \geq 1$.

Theorem (MMS)

If α *has at most three parts then* $r(q;\alpha)$ *is rank symmetric, BI, or TI.*

The following recursion also has a version for s even.

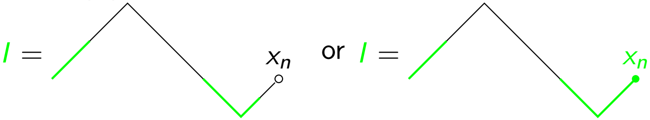
Lemma

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$. Then for s odd

$$r(q;\alpha) = r(q;\alpha_1,\ldots,\alpha_{s-1},\alpha_s - 1) + q^{\alpha_s+1} \cdot r(q;\alpha_1,\ldots,\alpha_{s-2},\alpha_{s-1}-1).$$

Proof.

If $I \in L(\alpha)$ then either $x_n \notin I$ or $x_n \in I$ where $n = \#F(\alpha)$. So



□

Using the lemma as well as induction on $\alpha_2 + \cdots + \alpha_s$:

Theorem

We have $r(q;\alpha)$ unimodal if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ satisfies

$$\alpha_1 \geq \alpha_2 + \alpha_3 + \cdots + \alpha_s.$$

A similar result hold when α_s plays the role of α_1.

For long segments other than the first or last we use:

Lemma

Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$, $n = \#F(\alpha)$, and for some t

$$\alpha_t \geq 1 + \sum_{i \neq t} \alpha_i. \tag{1}$$

Let S be the segment of length α_t, $F' = F - S$,

$$m = \#F' \quad \text{and} \quad \ell = \#L(F')$$

Then the maximum size of a rank of $L = L(\alpha)$ is ℓ and this maximum occurs at ranks $m+1$ through $n-m-1$.

Proof.

If $I \in L(\alpha)$ then $I = J \cup K$ where $J \in L(F')$ and $K \in L(S)$. Since $L(S)$ is a chain, given J and $\operatorname{rk} I$, there is ≤ 1 choice for K. □

This lemma permits us to prove the two long segment results from the previous section.

A *chain decomposition (CD)* of a poset P is a partition of P into disjoint saturated chains. If P is ranked then the *center* of a chain C is

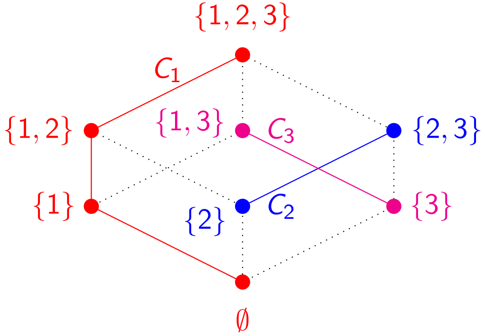
$$\text{cen } C = \frac{\mathrm{rk}(\min C) + \mathrm{rk}(\max C)}{2}.$$

A CD is *symmetric (SCD)* if for all chains C in the CD

$$\text{cen } C = \frac{n}{2} \text{ where } n = \max_{x \in P}(\mathrm{rk}\, x).$$

If P has an SCD then $r(q; P)$ is rank symmetric and unimodal.

Ex. B_3



$\text{cen } C_1 = \frac{0+3}{2} = \frac{3}{2}$

$\text{cen } C_2 = \frac{1+2}{2} = \frac{3}{2}$

$\text{cen } C_3 = \frac{1+2}{2} = \frac{3}{2}$

A CD is *top centered (TCD)* if for all chains C in the CD

$$\text{cen}\, C = \frac{n}{2} \quad \text{or} \quad \frac{n+1}{2}.$$

A *bottom centered CD (BCD)* has cen $C = n/2$ or $(n-1)/2$ for all chains C. If P has a TCD or BCD then its rank sequence is top or bottom interlacing, respectively.

Conjecture (MSS)

For any α, the lattice $L(\alpha)$ admits an SCD, TCD, or BCD consistent with the interlacing conjecture.

Theorem (MSS)

The previous conjecture is true if

1. *α has at most three segments.*
2. *$\alpha = (d, 1, d, 1, d, \ldots)$ for some d.*
3. *Under the hypotheses of the inductive long segment result.*

□

Let P be a poset on $[n] = \{1, 2, \ldots, n\}$. Construct the chains $C_1, C_2, C_3, \ldots$ of a CD of $L = L(P)$ as follows. Suppose $C_1, \ldots, C_{i-1}$ have been constructed. Since $P = [n]$ as sets, we can consider any ideal I of P as a subset of $\{1, \ldots, n\}$. So given two ideals, we can compare them in the lexicographic order on subsets. Now form C_i by starting with the unique ideal I_0 which has minimum rank and is also lexicographically least in

$$L' = L - (C_1 \cup \cdots \cup C_{i-1}).$$

Consider all ideals of L' covering I_0 and take the lexicographically least of them to be the next element I_1 on C_i. We continue in this manner until we come to an ideal which has no cover in L' at which point C_i terminates. We have the following conjecture which we have verified for all compositions α with $\sum_i \alpha_i \leq 6$.

Conjecture

For every α there is a labeling of $F(\alpha)$ with $[n]$ such that the correponding lexicographic CD of $L(\alpha)$ is an SCD, TCD, or BCD.

Explicit formula. There is an explicit formula for $r(q; \alpha)$ as a sum of products of powers of q and q-integers $[n]_q$. But it doesn't seem to be useful for more than 4 segments.

Partial symmetry. All fences with an odd number of segments are symmetric at the ends of the distribution.

Theorem (Elizalde and S)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ *where* s *is odd and supose* $\#F(\alpha) = n$*. If* $k \leq \min(\alpha_1, \alpha_s)$ *then*

$$a_k = a_{n-k}$$

□

THANKS FOR LISTENING!