Rooted partitions and number-theoretic functions

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Introduction

An identity with ϕ

An identity with μ

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and if $n \in \mathbb{N}$ then let $[n] = \{1, 2, \ldots, n\}$. A *partition* of $n \in \mathbb{N}$, written $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_m)$ with $\sum_i \lambda_i = n$. Let

$$\mathcal{P}(n) = \{\lambda \mid \lambda \vdash n\} \text{ and } p(n) = \#\mathcal{P}(n)$$

where $\#$ denotes cardinality. We write $|\lambda| := \sum_i \lambda_i$.
Ex. $\mathcal{P}(4) = \{(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)\}$ so $p(4) = 5$.

For $n \geq 1$, *Euler's totient function* is $\phi(n)$ where

$$\Phi(n) = \{k \in [n] \mid \gcd(k, n) = 1\} \text{ and } \phi(n) = \#\Phi(n).$$

Ex. $\Phi(12) = \{1, 5, 7, 11\}$ so $\phi(12) = 4$.

Finally, still for $n \geq 1$, the *Möbius function* is

$$\mu(n) = \begin{cases} (-1)^{\delta(n)} & \text{if } n \text{ is square free,} \\ 0 & \text{else,} \end{cases}$$

where $\delta(n)$ is number of distinct prime divisors of n.
Ex. $\mu(70) = \mu(2 \cdot 5 \cdot 7) = (-1)^3 = -1$ but $\mu(50) = \mu(2 \cdot 5^2) = 0$.

Let

$$S_k(n) = \text{ number of } k\text{'s in all the } \lambda \vdash n.$$

Ex. If $n = 4$ and $k = 1$ then

$$\mathcal{P}(4) = \{(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)\}$$

and, counting the number of ones in each partition,

$$S_1(4) = 0 + 1 + 0 + 2 + 4 = 7.$$

Let

$$S_k^{\geq r}(n) = \text{ number of } k\text{'s in all the } \lambda \vdash n \text{ with parts} \geq r.$$

Merca and Schmidt prove the following identities mainly by manipulation of q-series. We prove them combinatorially.

Theorem (Merca-Schmidt)

1. $S_1(n) = \sum_{k=2}^{n+1} \phi(k)\ S_k^{\geq 2}(n+1)$.
2. $p(n) = \sum_{k=3}^{n+3} \frac{\phi(k)}{2}\ S_k^{\geq 3}(n+3)$.
3. $p(n) = \sum_{k=1}^{n+1} \mu(k)\ S_k(n+1)$.
4. $p(n) = -\sum_{k=2}^{n+2} \mu(k)\ S_k^{\geq 2}(n+2)$.

Call a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n *rooted* if one of its parts, say one of the k's, has been distinguished. This part is called the *root* and will be denoted $\hat{k}$.

Ex. If $\lambda = (5, 2, 2, 2, 1, 1)$ then the ways to root λ at 2 are

$$(5, \hat{2}, 2, 2, 1, 1),\ (5, 2, \hat{2}, 2, 1, 1),\ \text{and } (5, 2, 2, \hat{2}, 1, 1).$$

Let

$$\mathcal{S}_k^{\geq r}(n) = \{\lambda \mid \lambda \vdash n \text{ rooted at } k \text{ and with parts} \geq r\}.$$

and $\mathcal{S}_k(n) = \mathcal{S}_k^{\geq 1}(n)$. Clearly $\#\mathcal{S}_k^{\geq r}(n) = S_k^{\geq r}(n)$ for all n, k, r.

Let λ, ν be two partitions with at most one of them rooted. Their *direct sum* $\lambda \oplus \nu$ is obtained by, for each k, concatenating the string of k's in λ with the string of k's in ν, including the $\hat{k}$ if one exists.

Ex. $(5, 2, 2, 1) \oplus (4, 4, 2, \hat{2}, 2, 1, 1) = (5, 4, 4, 2, 2, 2, \hat{2}, 2, 1, 1, 1)$.
Note that this operation is not commutative as

$$(4, 4, 2, \hat{2}, 2, 1, 1) \oplus (5, 2, 2, 1) = (5, 4, 4, 2, \hat{2}, 2, 2, 2, 1, 1, 1).$$

Theorem (Merca-Schmidt) $S_1(n) = \sum_{k=2}^{n+1} \phi(k)\ S_k^{\geq 2}(n+1).$

Proof. (Sagan) We give a bijection $\mathcal{S}_1(n) \to \mathcal{S}'(n+1)$ where

$$\mathcal{S}_1(n) = \{\lambda \mid \lambda \vdash n \text{ rooted at } 1\}$$

$$\mathcal{S}'(n+1) = \{(\lambda', r) \mid \lambda' \in \mathcal{S}_k^{\geq 2}(n+1) \text{ for some } k \text{ and } r \in \Phi(k)\}.$$

Given $\lambda \in \mathcal{S}_1(n)$, let

$$o = \text{ number of 1's in } \lambda,$$

$$p = \text{ position of } \hat{1} \text{ (positions numbered left to right)},$$

$$g = \gcd(o+1, p).$$

Ex. Suppose that

$$\lambda = (4, 4, 2, 1, 1, \hat{1}, 1, 1) \in \mathcal{S}_1(15).$$

So

$$o = 5,$$

$$p = 3,$$

$$g = \gcd(5+1, 3) = 3.$$

Write

$$\lambda = \nu \oplus \omega \text{ where } \omega \text{ contains all the 1's and } \hat{1},$$

Let

$$\lambda' = \nu \oplus \omega' \text{ where } \omega' = (\overbrace{\widehat{(o+1)}/g, (o+1)/g, \ldots, (o+1)/g}^{g}),$$
$$r = p/g.$$

Ex. We have

$$\lambda = (4,4,2,1,1,\hat{1},1,1) = (4,4,2) \oplus (1,1,\hat{1},1,1).$$

Recall $o = 5$, $p = 3$, and $g = \gcd(5+1,3) = 3$. Let

$$\omega' = \widehat{(5+1)}/3, (5+1)/3, (5+1)/3) = (\hat{2},2,2).$$

So $\lambda' = (4,4,2) \oplus (\hat{2},2,2) = (4,4,2,\hat{2},2,2)$ and $r = 3/3 = 1$.

One can show that the map $\lambda \mapsto (\lambda', r)$ is a bijection by constructing its inverse. □

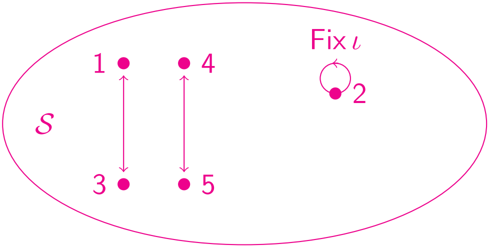
Let $\mathcal{S}$ be a finite set. Bijection $\iota : \mathcal{S} \to \mathcal{S}$ is an *involution* if $\iota^2 = \mathrm{id}$, the identity map. Any bijection $\iota : \mathcal{S} \to \mathcal{S}$ can be considered as a digraph with vertex set $\mathcal{S}$ and an arc $\vec{st}$ if $\iota(s) = t$. This graph can be decomposed into directed cycles.

Lemma ι is an involution iff each cycle contains 1 or 2 elements.

Let

$$\mathrm{Fix}\,\iota = \{s \in \mathcal{S} \mid \iota(s) = s\}.$$

Ex. Let $\mathcal{S} = [5]$ and $\iota(1) = 3, \iota(2) = 2, \iota(3) = 1, \iota(4) = 5, \iota(5) = 4$.
Then $\iota^2(1) = \iota(3) = 1$ and similarly $\iota^2(s) = s$ for all $s \in [5]$.
The cycle containing 1 is $1 \leftrightarrow \iota(1)$ or $1 \leftrightarrow 3$.
Also $\mathrm{Fix}\,\iota = \{2\}$.

$\iota =$ 

A set $\mathcal{S}$ is *signed* if there is a map $\operatorname{sgn} : \mathcal{S} \to \{-1, +1\}$. Let

$$\mathcal{S}^+ = \{s \in \mathcal{S} \mid \operatorname{sgn} s = +1\}, \qquad \mathcal{S}^- = \{s \in \mathcal{S} \mid \operatorname{sgn} s = -1\}.$$

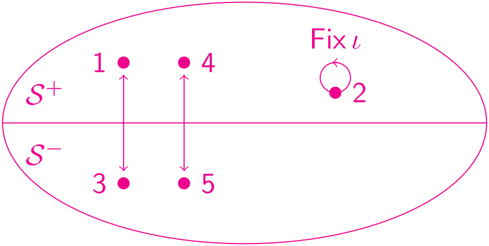
Involution $\iota : \mathcal{S} \to \mathcal{S}$ is *sign reversing* if

1. For every two-cycle $s \leftrightarrow t$ of ι we have $\operatorname{sgn} s = -\operatorname{sgn} t$.
2. For every fixed point s of ι we have $\operatorname{sgn} s = +1$.

So in this case

$$\sum_{s \in \mathcal{S}} \operatorname{sgn} s = \# \operatorname{Fix} \iota.$$

Ex. Let $\operatorname{sgn} 1 = \operatorname{sgn} 2 = \operatorname{sgn} 4 = +1$ and $\operatorname{sgn} 3 = \operatorname{sgn} 5 = -1$.

$\iota =$ 

Theorem (Merca-Schmidt) $p(n) = \sum_{k=1}^{n+1} \mu(k)\, S_k(n+1)$.

Proof. (Sagan) By definition of μ we can restrict the sum to square-free k. Let

$\mathcal{S}(n+1) = \{\lambda \vdash n+1 \mid \lambda$ is a partition rooted at a square-free part$\}$.

Let the sign of a partition λ with root $\hat{k}$ be

$$\operatorname{sgn} \lambda = \mu(k) = (-1)^{\delta(k)}.$$

Since the number of ways to root λ at k is the number of k's in λ

$$\sum_{\lambda \in \mathcal{S}(n+1)} \operatorname{sgn} \lambda = \sum_{k \text{ square-free}} \sum_{\lambda \in \mathcal{S}_k(n+1)} \mu(k) = \sum_{k \text{ square-free}} \mu(k)\, S_k(n+1).$$

Also, there is a bijection between partitions $\nu \in \mathcal{P}(n)$ and the partitions $\nu' \in \mathcal{S}(n+1)$ obtained by inserting a $\hat{1}$ at the end of ν.

Ex. $\nu = (5, 3, 3, 2, 1, 1) \leftrightarrow \nu' = (5, 3, 3, 2, 1, 1, \hat{1})$.

Note

$$\operatorname{sgn} \nu' = \mu(1) = 1 > 0.$$

So it suffices to produce a sign-reversing involution ι on $\mathcal{S}(n+1)$ with the rooted partitions ending in $\hat{1}$ as fixed points.

To construct the sign-reversion involution, we will need

$$\pi(n) = \begin{cases} \text{smallest prime dividing } n & \text{if } n \geq 2, \\ \infty & \text{if } n = 1, \end{cases}$$

where we consider $\infty > p$ for any prime p.

Ex. $\pi(75) = \pi(3 \cdot 5^2) = 3$ and $\pi(1) = \infty$.

If $\lambda \in \mathcal{S}(n+1)$ with root $\hat{k}$ then let m be the number of parts equal to k after and including $\hat{k}$.

Write

$$\lambda = \nu \oplus \kappa \text{ where } \kappa = (\overbrace{\hat{k}, k, \ldots, k}^{m}).$$

Ex. $\lambda = (3, 3, 2, \hat{2}, 2, 2, 1, 1)$. Thus the root is $k = 2$ and there are $m = 3$ parts of that size after and including $\hat{2}$. Furthermore

$$\lambda = (3, 3, 2, 1, 1) \oplus (\hat{2}, 2, 2)$$

We now have 2 cases for constructing $\lambda' = \iota(\lambda)$ depending on the the relative sizes of $\pi(k)$ and $\pi(m)$. Consider any $\lambda \in \mathcal{S}(n+1)$ not ending with $\hat{1}$. So $\min\{\pi(k), \pi(m)\} \neq \infty$ making both cases well defined.

Case 1: $\pi(k) \leq \pi(m)$. Then we let

$$k_1 = k/\pi(k) \quad \text{and} \quad m_1 = m \cdot \pi(k).$$

Also let

$$\lambda' = \nu \oplus \kappa' \text{ where } \kappa' = (\overbrace{\widehat{k_1}, k_1, \ldots, k_1}^{m_1}).$$

Ex. $\lambda = (3,3,2,1,1) \oplus (\hat{2},2,2)$, with root $k = 2$ and $m = 3$ parts in $\kappa = (\hat{2},2,2)$. Now $\pi(k) = \pi(2) = 2$ and $\pi(m) = \pi(3) = 3$ so $\pi(k) \leq \pi(m)$. Let

$$k_1 = k/\pi(k) = 2/2 = 1 \text{ and } m_1 = m \cdot \pi(k) = 3 \cdot 2 = 6.$$

So $\kappa' = (\hat{1},1,1,1,1,1)$ and

$$\lambda' = (3,3,2,1,1) \oplus (\hat{1},1,1,1,1,1) = (3,3,2,1,1,\hat{1},1,1,1,1,1).$$

Case 2: $\pi(k) > \pi(m)$. Then we let

$$k_2 = k \cdot \pi(m) \quad \text{and} \quad m_2 = m/\pi(m).$$

Also let

$$\lambda' = \nu \oplus \kappa'' \text{ where } \kappa'' = (\overbrace{\widehat{k_2}, k_2, \ldots, k_2}^{m_2}).$$

One can check that Cases 1 and 2 are sign-reversing inverses. $\square$

References

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   Click the link for the book on my web page for a free copy.
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THANKS FOR LISTENING!