Rationality, irrationality, and Wilf equivalence in generalized factor order

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Outline

Let P be a set and consider the corresponding *free monoid*

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Theorem

Suppose $|P|$ is finite. Then a language $\mathcal{L} \subseteq P^*$ is regular iff $\mathcal{L} = \mathcal{L}(\Delta)$ for some NFA Δ. ■

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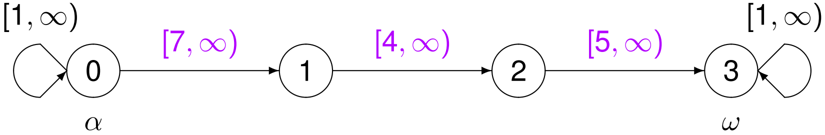

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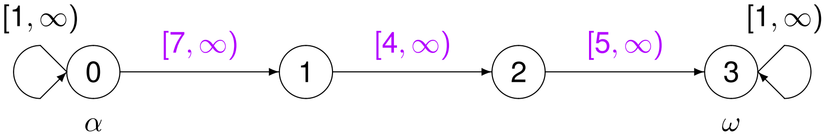

So $745 \leq 968864$ corresponding to the NFA path

$0 \xrightarrow{9} 0 \xrightarrow{6} 0 \xrightarrow{8} 1 \xrightarrow{8} 2 \xrightarrow{6} 3 \xrightarrow{4} 3$.

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If $P = \mathbb{P}$ and $u \in \mathbb{P}^*$ then one can define the following generating function which is a specialization of $F(u)$:

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There are no more Wilf equivalences since

$$F(123; t, x) = \frac{t^3x^6(1 - x + tx)}{(1 - x)^3(1 - x - tx + tx^3 - t^2x^4)}$$

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Outline

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1. $|uv| \leq n$ *and* $|v| \geq 1$,
2. $uv^i w \in \mathcal{L}$ *for all* $i \geq 1$. ■

Theorem

In ordinary factor order with $P = \{a, b\}$, $\mathcal{M}(a)$ is not regular.

Proof By contradiction: let n be the Pumping Lemma constant. Choose $z = ab^n ab^n a$. Then $z \in \mathcal{M}(a)$: $o(z) = ab^n a$ and $i(z) = b^n ab^n$ giving $a \leq o(z) \not\leq i(z)$. So

$$\mu(a, ab^n ab^n a) = \mu(a, ab^n a) = \mu(a, a) = 1.$$

Now pick any prefix uv of z as in the Pumping Lemma. Suppose $u \neq \epsilon$ ($u = \epsilon$ is similar). So $v = b^j$ for some j with $1 \leq j < n$. Let $i = 2$ with corresponding $z' = uv^2 w = ab^{n+j} ab^n a$. But $o(z') = a$ and $i(z') = b^{n+j} ab^n$.

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Outline

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3. What is $\mu(u, w)$ in generalized factor order?