Rowmotion on fences

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Fences

Rowmotion on fences and tilings

Self-dual posets

Comments and future work

Let G be a finite group acting on a finite set S. Let $\mathbb{N}$ be the nonnegative integers and $\operatorname{st} : S \to \mathbb{N}$ be a statistic. If $\mathcal{O} \subseteq S$ then we let

$$\operatorname{st} \mathcal{O} = \sum_{x \in S} \operatorname{st} x.$$

Call st *homomesic* if $\operatorname{st} \mathcal{O} / \# \mathcal{O}$ is constant over all orbits $\mathcal{O}$ where the hash tag is cardinality. Call st *homometric* if for any two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we have

$$\# \mathcal{O}_1 = \# \mathcal{O}_2 \implies \operatorname{st} \mathcal{O}_1 = \operatorname{st} \mathcal{O}_2.$$

Note that homomesy implies homometry, but not conversely.

Let $(P, \trianglelefteq)$ be a finite poset, $\mathcal{A}(P)$ be the set of antichains of P, and $\mathcal{I}(P)$ be the set of (lower order) ideals of P. Let $\rho : \mathcal{A}(P) \to \mathcal{A}(P)$ be rowmotion on antichains, so $\rho(A)$ is the set of minimal elements of the complement of the ideal generated by A. Let $\hat{\rho} : \mathcal{I}(P) \to \mathcal{I}(P)$ be rowmotion on ideals.

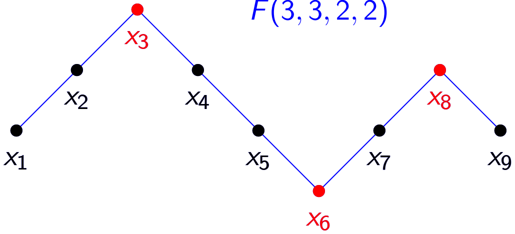
A *fence* is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$ and covers

$$x_1 \lhd x_2 \lhd \ldots \lhd x_a \rhd x_{a+1} \rhd \ldots \rhd x_b \lhd x_{b+1} \lhd \cdots$$

where $a, b, \ldots$ are positive integers.

Ex. $\check{F}(3,3,2,2)$

$F =$



4 segments

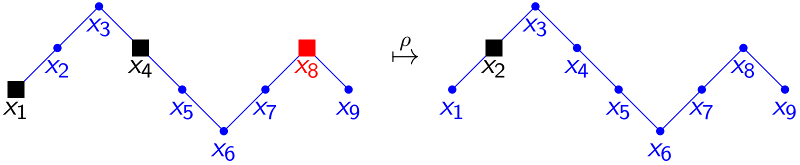
shared

unshared

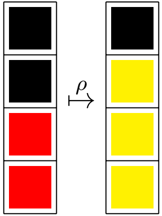
Fences have important connections with cluster algebras, q-analogues, unimodality, and Young diagrams. The maximal chains of F are called *segments*. Elements on two segments are called *shared*. All other elements are *unshared*. If F has s segments then we let $F = \check{F}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ where for all i

$$\alpha_i = (\#\text{ of unshared elements on segment } i) + 1.$$

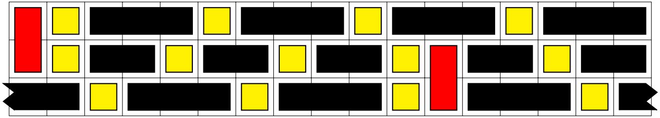
As an example of rowmotion on a fence F, consider the fence below and $A = \{x_1, x_4, x_8\}$ indicated by squares. So $\rho(A) = \{x_2\}$.



Represent an antichain $A \subset F$ using a column of 4 boxes, with the box in row i from the top corresponding to the ith segment S_i from the left. We color the box for S_i by black if $S_i \cap A$ is an unshared element, red if $S_i \cap A$ is a shared element, or yellow if $S_i \cap A = \emptyset$.



Pasting together such colored columns, we can model any orbit of ρ on a fence $F = \check{F}(\alpha_1, \ldots, \alpha_s)$ as a tiling of a cylinder C_s of boxes having s rows. One of the orbits in $\check{F}(4, 3, 4)$ has the following tiling where the left and right ends of the rectangle are identified.



We can characterize these tilings as follows. If $\alpha = (\alpha_1, \ldots, \alpha_s)$, then an *$\alpha$-tiling* is a tiling of C_s using yellow 1×1 tiles, red 2×1 tiles, and black $1 \times (\alpha_i - 1)$ tiles in row i, for $1 \leq i \leq s$, such that the following hold for all rows.

(a) If $\alpha_i \geq 2$ and the red tiles are ignored, then the black and yellow tiles alternate in row i.

(b) There is a red tile in a column covering rows i and $i + 1$ if and only if either the next column contains two yellow tiles in those two rows when i is odd, or the previous column contains two yellow tiles in those two rows when i is even.

$b_i :=$ the number of black tiles in row i of a tiling,

$r_i :=$ the number of red tiles whose top box is in row i of a tiling,

$\chi(\mathcal{O}) :=$ the number of antichain elements in orbit $\mathcal{O}$.

Lemma (EPRS)

Given an orbit $\mathcal{O}$ in fence $\check{F}(\alpha)$ with corresponding α-tiling

$$\chi(\mathcal{O}) = \sum_{i=1}^{s}(b_i\alpha_i - b_i + r_i).$$

One can also compute χ_x, the number of times a given element x appears in an orbit, and derive corresponding results for ideals.

Theorem (EPRS)

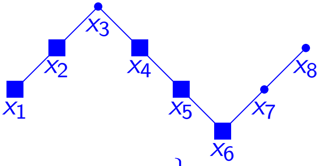
1. *If x is unshared and y, z are the shared elements on the same segment S_i then $\alpha_i\chi_x + \chi_y + \chi_z$ is 1-mesic.*
2. *For $\check{F}(a, b)$ all orbits $\mathcal{O}$ have size $\ell = \mathrm{lcm}(a, b)$ except one $\mathcal{O}'$ of size $\ell + 1$. For the orbits of size ℓ we have $\chi(\mathcal{O}) = \frac{2ab-a-b}{\mathrm{gcd}(a,b)} := m$. For the other orbit $\chi(\mathcal{O}') = m + 1$.*

Let P^* be the dual of poset P. Suppose P is self dual so that $P \cong P^*$. Thus there exists and order-reversing bijection $\kappa : P \to P$. Define the *ideal complement* of $I \in \mathcal{I}(P)$ as

$$\bar{I} = c \circ \kappa(I)$$

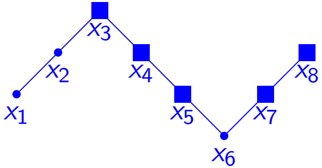
where $c(S) = P - S$ for any $S \subseteq P$. Note that $\#I + \#\bar{I} = \#P$.

Ex. $\kappa(x_i) = x_{9-i}$

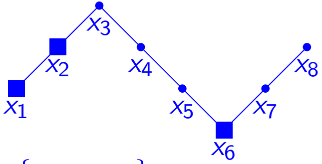


$I = \{x_1, x_2, x_4, x_5, x_6\}$

$\overset{\kappa}{\mapsto}$



$\overset{c}{\mapsto}$



$\bar{I} = \{x_1, x_2, x_6\}$

Let

$\hat{\chi}(\mathcal{O})$ = the number of ideal elements in an orbit $\mathcal{O}$ of $\hat{\rho}$.

Theorem

Let P be self-dual with $n = \#P$, and fix an order-reversing bijection $\kappa : P \to P$. Let $I \in \mathcal{I}(P)$.

1. *If $I, \overline{I} \in \mathcal{O}$ for some orbit $\mathcal{O}$, then*
$$\frac{\hat{\chi}(\mathcal{O})}{\#\mathcal{O}} = \frac{n}{2}.$$
2. *If $I \in \mathcal{O}$ and $\overline{I} \in \overline{\mathcal{O}}$ for some orbits $\mathcal{O}$ and $\overline{\mathcal{O}}$ with $\mathcal{O} \neq \overline{\mathcal{O}}$, then $\#\mathcal{O} = \#\overline{\mathcal{O}}$ and*
$$\frac{\hat{\chi}(\mathcal{O} \uplus \overline{\mathcal{O}})}{\#(\mathcal{O} \uplus \overline{\mathcal{O}})} = \frac{n}{2}.$$

Consider the group generated by the action of $\hat{\rho}$ and the map $I \mapsto \overline{I}$. The orbits of this action will be called *superorbits*.

Theorem

If P is self-dual with $n = \#P$ then $\hat{\chi}$ is $(n/2)$-mesic on superorbits.

Constant α. Let $\alpha = (a^s) = (\underbrace{a, \dots, a}_{s})$.

Conjecture

Let $F = \check{F}(a^s)$.

1. *The statistic χ is homometric.*
2. *If s is odd then the statistic $\hat{\chi}$ is $n/2$-mesic where $n = \#F$.*

Toggles. Let S be a set and let $\mathcal{T}$ be the toggle group associated with some family of subsets of S. The *base graph* of $\mathcal{T}$, denoted $G_{\mathcal{T}}$, has as vertices the toggles τ_x for $x \in S$, and it has an edge $\tau_x\tau_y$ if τ_x and τ_y do not commute.

Theorem

Let $G_{\mathcal{T}}$ be acyclic and w, w' be any two Coxeter elements of $\mathcal{T}$ generating groups W, W'. Then any linear combination of indicator functions χ_y for $y \in S$ is c-mesic or homometric under the action of W if and only if it is c-mesic or homometric, respectively, under the action of W'.

In particular, the base graph of the toggle group for the ideals of a fence have an acyclic base graph, so this theorem applies.

THANKS FOR LISTENING!