Rowmotion on fences

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Let G be a finite group acting on a finite set S. Let $\mathbb{N}$ be the nonnegative integers and $\operatorname{st} : S \to \mathbb{N}$ be a statistic. If $\mathcal{O} \subseteq S$ then we let

$$\operatorname{st} \mathcal{O} = \sum_{x \in \mathcal{O}} \operatorname{st} x.$$

Call st *homomesic* if $\operatorname{st} \mathcal{O} / \# \mathcal{O}$ is constant over all orbits $\mathcal{O}$ where the hash tag is cardinality. In particular, st is *c-mesic* if, for all orbits $\mathcal{O}$,

$$\frac{\operatorname{st} \mathcal{O}}{\# \mathcal{O}} = c.$$

$$S_{n,k} := \{w_1 w_2 \ldots w_n \mid w_i \in \{0,1\} \text{ for all } i, \text{ and having } k \text{ ones}\}$$

with rotation $w_1 w_2 \ldots w_n \mapsto w_n w_1 \ldots w_{n-1}$, and *inversion statistic*

$$\mathrm{inv}\, w_1 w_2 \ldots w_n = \#\{(i,j) \mid i < j \text{ and } w_i > w_j\}.$$

Theorem (Propp-Roby)

The inversion statistic is $k(n-k)/2$-mesic for rotation on $S_{n,k}$.

Call st *homometric* if for any two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we have

$$\#\mathcal{O}_1 = \#\mathcal{O}_2 \implies \mathrm{st}\,\mathcal{O}_1 = \mathrm{st}\,\mathcal{O}_2.$$

Note that homomesy implies homometry, but not conversely.

Ex. When $n = 4$ and $k = 2$ there are two orbits

w	$\mathrm{inv}\, w$	w	$\mathrm{inv}\, w$
1100	4	1010	3
0110	2	0101	1
0011	0		
1001	2		
average $= 8/4 = 2$		average $= 4/2 = 2$	

Let $(P, \trianglelefteq)$ be a finite poset. The sets of *antichains*, *(lower) ideals*, and *upper ideals* of P are

$$\mathcal{A}(P) = \{A \subseteq P \mid \text{no two elements of } A \text{ are comparable}\},$$
$$\mathcal{I}(P) = \{I \subseteq P \mid x \in I \text{ and } y \leq x \text{ implies } y \in I\},$$
$$\mathcal{U}(P) = \{U \subseteq P \mid x \in U \text{ and } y \geq x \text{ implies } y \in U\}.$$

An ideal produces an antichain via $\Delta : \mathcal{I}(P) \to \mathcal{A}(P)$ where

$$\Delta(I) = \{x \in P \mid x \text{ is a maximal element of } I\}.$$

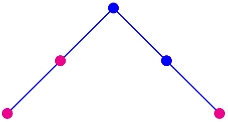
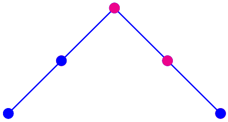
An upper ideal produces an antichain via $\nabla : \mathcal{U}(P) \to \mathcal{A}(P)$ where

$$\nabla(U) = \{x \in P \mid x \text{ is a minimal element of } U\}.$$

Ideals produce upper ideals via $c : \mathcal{I}(P) \to \mathcal{U}(P)$ where

$$c(I) = P - I.$$

Ex.

$I = \quad U =$  $\overset{\Delta}{\mapsto} \quad \overset{\nabla}{\mapsto}$ $A = \quad U =$ 

Rowmotion on antichains of poset P is $\rho : \mathcal{A}(P) \to \mathcal{A}(P)$ where

$$A \overset{\Delta^{-1}}{\mapsto} I \overset{c}{\mapsto} U \overset{\nabla}{\mapsto} \rho(A).$$

Rowmotion on antichains was first studied by Duchet (in a special case) and independently by Brouwer and Schrijver. *Rowmotion on ideals* of poset P is $\hat{\rho} : \mathcal{I}(P) \to \mathcal{I}(P)$ where

$$I \overset{c}{\mapsto} U \overset{\nabla}{\mapsto} A \overset{\Delta^{-1}}{\mapsto} \hat{\rho}(I).$$

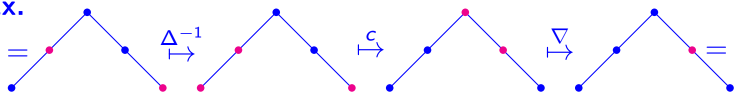
We will study two statistics. For antichains $A \in \mathcal{A}(P)$ define

$$\chi(A) = \#A$$

where the hash symbol is cardinality. For ideals $I \in \mathcal{I}(P)$ define

$$\hat{\chi}(I) = \#I.$$

Ex.

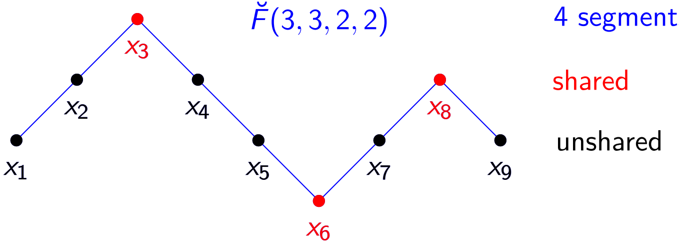
$A =$  $= \rho(A)$

A *fence* is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$ and covers

$$x_1 \lhd x_2 \lhd \ldots \lhd x_a \rhd x_{a+1} \rhd \ldots \rhd x_b \lhd x_{b+1} \lhd \cdots$$

where $a, b, \ldots$ are positive integers.

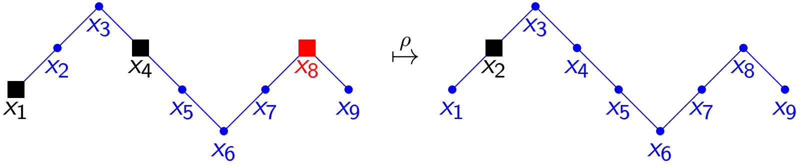
Ex. $\check{F}(3,3,2,2)$ 4 segments

$F =$ 

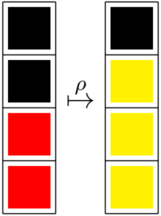
Fences have important connections with cluster algebras, q-analogues, unimodality, and Young diagrams. The maximal chains of F are called *segments*. Elements on two segments are called *shared*. All other elements are *unshared*. If F has s segments then we let $F = \check{F}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ where for all i

$$\alpha_i = (\# \text{ of unshared elements on segment } i) + 1.$$

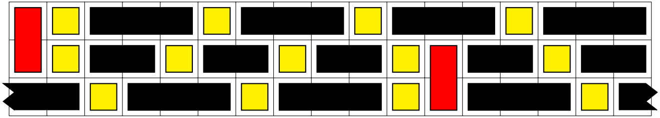
As an example of rowmotion on antichains in a fence, consider F below and $A = \{x_1, x_4, x_8\}$ indicated by squares. So $\rho(A) = \{x_2\}$.



Represent an antichain $A \subset F$ using a column of 4 boxes, with the box in row i from the top corresponding to the ith segment S_i from the left. We color the box for S_i by black if $S_i \cap A$ is an unshared element, red if $S_i \cap A$ is a shared element, or yellow if $S_i \cap A = \emptyset$.



Pasting together such colored columns, we can model any orbit of ρ on a fence $F = \check{F}(\alpha_1, \ldots, \alpha_s)$ as a tiling of a cylinder C_s of boxes having s rows. One of the orbits in $\check{F}(4, 3, 4)$ has the following tiling where the left and right ends of the rectangle are identified.



We can characterize these tilings as follows. If $\alpha = (\alpha_1, \ldots, \alpha_s)$, then an *$\alpha$-tiling* is a tiling of C_s using yellow 1×1 tiles, red 2×1 tiles, and black $1 \times (\alpha_i - 1)$ tiles in row i, for $1 \leq i \leq s$, such that the following hold for all rows.

(a) If $\alpha_i \geq 2$ and the red tiles are ignored, then the black and yellow tiles alternate in row i.

(b) There is a red tile in a column covering rows i and $i + 1$ if and only if either the next column contains two yellow tiles in those two rows when i is odd, or the previous column contains two yellow tiles in those two rows when i is even.

$$b_i := \text{the number of black tiles in row } i \text{ of a tiling,}$$
$$r_i := \text{the number of red tiles with top box in row } i \text{ of a tiling,}$$
$$\chi(\mathcal{O}) := \text{the number of antichain elements in orbit } \mathcal{O}.$$

Lemma (EPRS)

Given an orbit $\mathcal{O}$ in fence $\check{F}(\alpha)$ with corresponding α-tiling

$$\chi(\mathcal{O}) = \sum_{i=1}^{s}(b_i\alpha_i - b_i + r_i).$$

One can also compute χ_x, the number of times a given element x appears in an orbit, and derive corresponding results for ideals.

Theorem (EPRS)

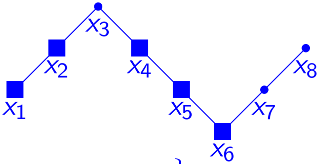
1. *If x is unshared and y, z are the shared elements on the same segment S_i then $\alpha_i\chi_x + \chi_y + \chi_z$ is 1-mesic.*
2. *For $\check{F}(a, b)$ all orbits $\mathcal{O}$ have size $\ell = \mathrm{lcm}(a, b)$ except one $\mathcal{O}'$ of size $\ell + 1$. For the orbits of size ℓ we have $\chi(\mathcal{O}) = \frac{2ab-a-b}{\gcd(a,b)} := m$. For the other orbit $\chi(\mathcal{O}') = m + 1$.*

Let P^* be the dual of poset P. Suppose P is self dual so that $P \cong P^*$. Thus there exists and order-reversing bijection $\kappa : P \to P$. Define the *ideal complement* of $I \in \mathcal{I}(P)$ as

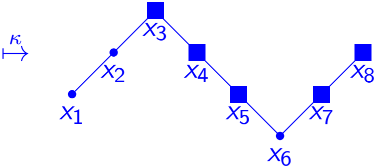
$$\bar{I} = c \circ \kappa(I)$$

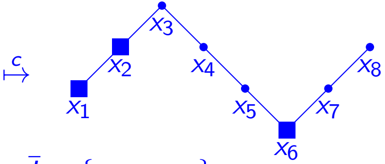
where $c(S) = P - S$ for any $S \subseteq P$. Note that $\#I + \#\bar{I} = \#P$.

Ex. $\kappa(x_i) = x_{9-i}$



$I = \{x_1, x_2, x_4, x_5, x_6\}$





$\bar{I} = \{x_1, x_2, x_6\}$

Let

$$\hat{\chi}(\mathcal{O}) = \text{the number of ideal elements in an orbit } \mathcal{O} \text{ of } \hat{\rho}.$$

Theorem (EPRS)

Let P be self-dual with $n = \#P$, and fix an order-reversing bijection $\kappa : P \to P$. Let $I \in \mathcal{I}(P)$.

1. *If $I, \overline{I} \in \mathcal{O}$ for some orbit $\mathcal{O}$, then*
$$\frac{\hat{\chi}(\mathcal{O})}{\#\mathcal{O}} = \frac{n}{2}.$$
2. *If $I \in \mathcal{O}$ and $\overline{I} \in \overline{\mathcal{O}}$ for some orbits $\mathcal{O}$ and $\overline{\mathcal{O}}$ with $\mathcal{O} \neq \overline{\mathcal{O}}$, then $\#\mathcal{O} = \#\overline{\mathcal{O}}$ and*
$$\frac{\hat{\chi}(\mathcal{O} \uplus \overline{\mathcal{O}})}{\#(\mathcal{O} \uplus \overline{\mathcal{O}})} = \frac{n}{2}.$$

Consider the group generated by the action of $\hat{\rho}$ and the map $I \mapsto \overline{I}$. The orbits of this action will be called *dihedral orbits*.

Corollary (EPRS)

If P is self-dual with $n = \#P$ then $\hat{\chi}$ is $(n/2)$-mesic on dihedral orbits.

Constant α.

Let $\alpha = (a^s) = (\underbrace{a, \ldots, a}_{s})$.

Conjecture

If $F = \check{F}(a^s)$ with s odd then then the statistic $\hat{\chi}$ is $n/2$-mesic where $n = \#F$.

For $\hat{\chi}$ one can not use our results on self-dual posets since I and $\overline{I}$ are not always in the same orbit.
In $F = \check{F}(2^s)$ we also have that χ is homomesic for any s, but this fails for general a even for homometry. Sam Hopkins pointed out that this follows from results in our paper and also from work of Chan, Haddadan, Hopkins, and Moci on balanced Young diagrams.

Palindromic α.

Sequence $a_0, a_1, \ldots, a_n$ is *palindromic* if $a_k = a_{n-k}$ for all $0 \leq k \leq n$. Write $\chi_k = \chi_{x_k}$ and $\hat{\chi}_k = \hat{\chi}_{x_k}$.

Proposition (EPRS)

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ where $\alpha_i \geq 2$ for all i. Also let $F = \check{F}(\alpha)$ and $n = \#F$. Let α, the black tile sequence $b_1, b_2, \ldots, b_s$, and the red tile sequence $r_1, r_2, \ldots, r_{s-1}$ be all palindromic for all orbits.

(a) *For all k the statistic $\chi_k - \chi_{n-k+1}$ is 0-mesic.*

(b) *If s is odd, then for all k the statistic $\hat{\chi}_k + \hat{\chi}_{n-k+1}$ is 1-mesic.*

Question

Let $F = \check{F}(\alpha)$ with α palindromic. Find necessary and/or sufficient conditions on α for the black or the red tile sequences to be palindromic for all rowmotion orbits.

Rooted trees.

A poset T is a *rooted tree* if its Hasse diagram is a graph-theoretic tree with a unique minimal element. Dangwal, Kimble, Liang, Lou, S, and Stewart have shown that there is a tiling model for rooted trees and that it can be used to prove many homometry results. A fence can be characterized as a poset whose Hasse diagram is a path, but with any number of minimal elements.

Question

Are there nice homoetries for posets whose Hasse diagram is a tree with any number of minimal elements?

References

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TAKK FOR AT DU LYTTET!

THANKS FOR LISTENING!

DANKE FÜRS ZUHÖREN

MERCI POUR VOTRE ATTENTION!