Stirling numbers for complex reflection groups

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Let $[n] = \{1, 2, \ldots, n\}$. A *partition of $[n]$ into k blocks* is $\rho = B_1/\ldots/B_k$ were $[n] = \uplus_i B_i$ and $B_i \neq \emptyset$ for all i. The *Stirling numbers of the second kind* are

$$S(n,k) = \#\{\rho \mid \rho \text{ is a partition of } [n] \text{ into } k \text{ blocks}\}.$$

Ex. If $n = 3$ then

k	1	2	3
ρ	123	1/23, 12/3, 13/2	1/2/3
$S(3,k)$	1	3	1

Let $\mathfrak{S}_n$ denote the symmetric group of permutations π of $[n]$. The *Stirling numbers of the first kind* are

$$s(n,k) = (-1)^{n-k}\#\{\pi \mid \pi \in \mathfrak{S}_n \text{ has } k \text{ disjoint cycles}\}.$$

Ex. If $n = 3$ then

k	1	2	3
π	$(1,2,3)$, $(1,3,2)$	$(1)(2,3)$, $(1,2)(3)$, $(1,3)(2)$	$(1)(2)(3)$
$s(3,k)$	2	-3	1

Let $\mathbf{x}_n = \{x_1, \ldots, x_n\}$ be a set of commuting variables. The *degree* of a monomial $m = x_1^{k_1} \ldots x_n^{k_n}$ is $\deg m = \sum_i k_i$. Define *complete homogeneous symmetric fuctions* by

$$h_k(\mathbf{x}_n) = \sum_{\deg m = k} m.$$

Ex.

k	1	2	3
$h_{3-k}(\mathbf{x}_k)$	$h_2(\mathbf{x}_1) = x_1^2$	$h_1(\mathbf{x}_2) = x_1 + x_2$	$h_0(\mathbf{x}_3) = 1$
$h_{3-k}(1, \ldots, k)$	1	3	1

Proposition

We have $S(n,k) = h_{n-k}(1, 2, \ldots, k)$.

Proof. Induct on n using the recursions

$$S(n,k) = S(n-1, k-1) + kS(n-1, k)$$

and

$$h_k(\mathbf{x}_n) = h_k(\mathbf{x}_{n-1}) + x_n h_{k-1}(\mathbf{x}_n)$$

to get the result. □

Define *elementary symmetric functions* by

$$e_k(\mathbf{x}_n) = \sum_{\deg m = k, m \text{ square free}} m.$$

Ex.

k	1	2	3
$(-1)^{3-k} e_{3-k}(\mathbf{x}_2)$	$e_2(\mathbf{x}_2) = x_1 x_2$	$-e_1(\mathbf{x}_2) = -(x_1 + x_2)$	$e_0(\mathbf{x}_2) = 1$
$(-1)^{3-k} e_{3-k}(1, 2)$	2	-3	1

Proposition

We have $s(n,k) = (-1)^{n-k} e_{n-k}(1, 2, \dots, n-1)$. □

Let P be a finite poset with a unique minimum element $\hat{0}$, and a *rank function* where for $x \in P$

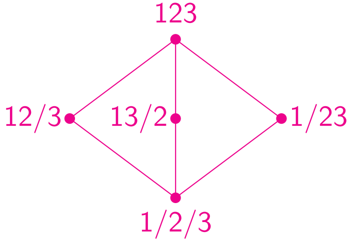
$$\operatorname{rk} x = \text{length of any maximal } \hat{0}\text{–}x \text{ chain.}$$

Let

$$\Pi_n = \text{set of partitions } \rho \text{ of } [n] \text{ ordered by refinement.}$$

Ex. if $n = 3$ then

$\Pi_3 =$



So if $\rho = B_1/\ldots/B_k \in \Pi_n$ then

$$\operatorname{rk} \rho = n - k.$$

The *Whitney numbers of the 2nd kind for P* are

$$W(P,k) = \sum_{\mathrm{rk}\, x = k} 1 = \#\{x \in P \mid \mathrm{rk}\, x = k\}.$$

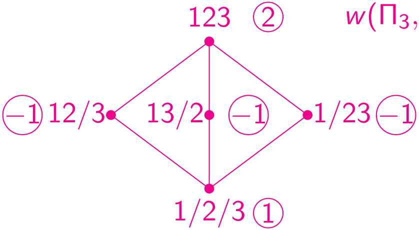
The *Möbius function of P* is defined by $\mu(\hat{0}) = 1$ and for $x > \hat{0}$

$$\mu(x) = -\sum_{y<x} \mu(y) \iff \sum_{y \le x} \mu(y) = \delta_{x,\hat{0}}.$$

The *Whiney numbers of the 1st kind for P* are

$$w(P,k) = \sum_{\mathrm{rk}\, x = k} \mu(x).$$

Ex. $W(\Pi_3, k):$ 1, 3, 1



$w(\Pi_3, k):$ 2, -3, 1

Proposition

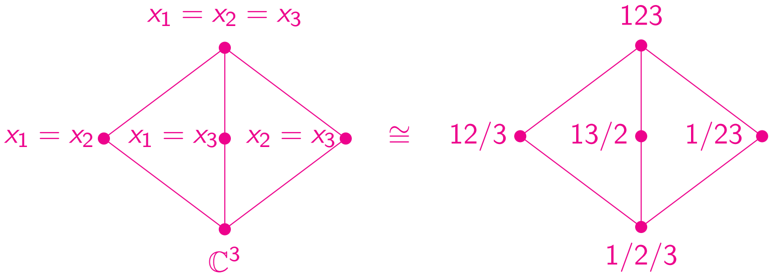
We have $W(\Pi_n, k) = S(n, n-k)$ and $w(\Pi_n, k) = s(n, n-k)$. □

A *hyperplane* in $\mathbb{C}^n$ is a subspace H with $\dim H = n-1$. A *hyperplane arrangement* is a finite set $\mathcal{A} = \{H_1, \ldots, H_k\}$ of hyperplanes. The *braid arrangement* in $\mathbb{C}^n$ is

$$Br_n = \{x_i = x_j \mid 1 \leq i < j \leq n\}.$$

The *intersection lattice* $L(\mathcal{A})$ of an arrangement is all subspaces $W \subseteq \mathbb{C}^n$ which can be obtained as the intersection of some of the hyperplanes in $\mathcal{A}$ ordered by reverse inclusion.

Ex. We have $Br_3 = \{x_1 = x_2,\ x_1 = x_3,\ x_2 = x_3\}$, with lattice



Proposition

We have $L(Br_n) \cong \Pi_n$ as posets. □

A *pseudoreflection* is a linear map $M : \mathbb{C}^n \to \mathbb{C}^n$ which fixes a hyperplane and is of finite order. A *complex reflection group* G is a group generated by pseudoreflections. Call G *irreducible* if its only G-invariant subspaces are $\mathbb{C}^n$ and the origin, and n is called G's *rank*. Shephard and Todd classified the finite irreducible complex reflection groups into 3 infinite families and 34 exceptionals.

$G(m,p,n) :=$ group of all $n \times n$ complex matrices M satisfying

1. Each row and column of M contains exactly one nonzero entry, say ζ_i in row i.
2. Each ζ_i is an mth root of unity.
3. We have $p|m$ and $(\zeta_1 \cdots \zeta_n)^{m/p} = 1$.

Ex. If

$$M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ ia \\ b \end{bmatrix} = \begin{bmatrix} a \\ ia \\ b \end{bmatrix}.$$

So M fixes $x_2 = ix_1$ and $M^2 = I$. Also $M \in G(4,p,3)$ for any $p|4$.

Note the following special cases:
$G(1,1,n)$: each $\zeta_i = 1$. So $G(1,1,n) =$ symmetric group, A_{n-1}.
$G(2,1,n)$: each $\zeta_i = \pm 1$. And $1 = (\zeta_1 \cdots \zeta_n)^{2/1}$ imposes no restriction. So $G(2,1,n) =$ hyperoctahedral group, B_n.
$G(2,2,n)$: each $\zeta_i = \pm 1$. And $1 = (\zeta_1 \cdots \zeta_n)^{2/2}$ implies an even number of -1 entries. So $G(2,2,n) = D_n$.
In general, given a finite complex reflection group G we let

$$\mathcal{A}(G) = \{H \mid H \text{ a fixed hyperplane of a pseudoreflection in } G\},$$
$$L(G) = \text{intersection lattice of } \mathcal{A}(G).$$

If G is irreducible of rank n then it's *Stirling numbers of the first and second kinds* are, respectively,

$$s(G,k) = w(L(G), n-k) \quad \text{and} \quad S(G,k) = W(L(G), n-k).$$

Theorem (Orlik-Solomon, 1980)

If G is a finite, irreducible complex reflection group with coexponents $e_1^*, \ldots, e_n^*$ then

$$s(G,k) = (-1)^{n-k} e_{n-k}(e_1^*, \ldots, e_n^*).$$

For $S(G,k)$ things are more complicated.

Lemma

The reflecting hyperplanes of $G(m,p,n)$ are of the form

1. $x_i = \zeta x_j$ *for* $\zeta^m = 1$ *and distinct* $i, j \in [n]$,
2. $x_i = 0$ *for* $i \in [n]$ *in the case* $p < m$.

Ex. In $G(4,1,3)$ the pseudoreflections

$$M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}$$

have corresponding hyperplanes

$$x_2 = ix_1 \quad \text{and} \quad x_3 = 0.$$

Theorem (SSS)

Let $G = G(m,p,n)$.

$$S(G,k) = \begin{cases} h_{n-k}(1, m+1, \ldots, km+1) := h(m,k,n) & \text{for } p < m, \\ h(m,k,n) - nh_{n-k-1}(m, 2m, \ldots, km) & \text{for } p = m. \end{cases}$$

Is there a way to interpret $S(G,k)$ in terms of partitions? Consider $G = G(2,1,n) = B_n$. The hyperplanes of B_n are of three types

1. $x_i = x_j$,
2. $x_i = -x_j$,
3. $x_k = 0$.

Corresponding partitions ρ of $\langle n \rangle = \{0, \pm 1, \ldots, \pm n\}$ will have

1. a block containing i, j and a different block containing $-i, -j$,
2. a block containing $i, -j$ and a different block containing $-i, j$,
3. the bock containing 0 also contains $\pm k$.

Ex. In $\mathbb{C}^5$ subspace $(x_1 = x_3 = -x_4) \cap (x_5 = 0)$ has partition

$$\rho = 0, -5, 5/1, 3, -4/-1, -3, 4/2/-2.$$

Partition $\rho = B_0/B_1/B_2/\ldots/B_{2m}$ of $\langle n \rangle$ is *type B_n* if

1. $0 \in B_0$, and if $k \in B_0$ then also $-k \in B_0$,
2. $B_{2i} = -B_{2i-1}$ for $i \geq 1$.

Theorem (Zaslavsky, 1982)

$S(B_n, m)$ is the number of type B_n partitions with $2m+1$ blocks.

To extend this to all $G(m,p,n)$ it is convenient to use digraphs. We assume $p < m$ for simplicity. A *weighted digraph* is a digraph Δ with vertices $V(\Delta)$, arcs $A(\Delta)$, and every arc a has a weight $\mathrm{wt}(a) \in \mathbb{C}$. Given $B \subseteq A$ we let $\mathrm{wt}\, B = \prod_{a \in B} \mathrm{wt}(a)$. Define a set of weighted digraphs $\mathcal{D}(m,p,n)$ which are those Δ satsifying:

1. $V(\Delta) = \{0, 1, \ldots, n\}$ and if $\overrightarrow{ij} \in A(\Delta)$ then $i < j$.
2. Each component of Δ is a tournament. The component containing 0 is denoted Δ_0 and called the *0-component*.
3. For each arc a we have

$$\mathrm{wt}(a) = \begin{cases} 0 & \text{if } a \in A(\Delta_0), \\ \text{an } m\text{th root of unity} & \text{else.} \end{cases}$$

4. Given any two vertices i, j and directed i–j paths P_1, P_2 we have $\mathrm{wt}\, P_1 = \mathrm{wt}\, P_2$.

Theorem (SSS)

If $G = G(m,p,n)$ then $S(G,k)$ is the number of $\Delta \in \mathcal{D}(m,p,n)$ having k components.

1. If not h **...** For some complex reflection groups (e.g., those corresponding to E_6, E_7, E_8), we have been unable to find a nice expression for $W(G, k)$ in terms of the $h_k(\mathbf{x}_n)$.

2. More on $w(G, k)$**.** If P is a ranked poset then the generating function for the $w(P, k)$ is called the *characteristic polynomial* $\chi(P; q)$. The Orlik-Solomon Theorem can be restated as saying that for G an irreducible complex reflection group, the roots of $\chi(L(G); q)$ are certain nonnegative integers called the coexponents of G. Various methods have been developed for showing that $\chi(P; q)$ factors over the nonnegative integers $\mathbb{N}$: signed graphs (Zaslavsky), free hyperplane arrangements (Saito and Terao), supersolvable lattices (Stanley), and quotient posets (Hallam and S). Certain subarrangements of Br_n can be shown to have characteristic polynomials which factor over $\mathbb{N}$ because they correspond to graphs having a perfect elimination order. What subarrangements of other $\mathcal{A}(G)$ have characteritic polynomials factoring over $\mathbb{N}$? Certain arrangements considered by Hallam, Martin, and S which have a definition similar to that of $\mathcal{D}(m, p, n)$ have this property.

THANKS FOR LISTENING!