Stirling numbers in type B

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Basic definitions

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We will use the notation

$$\begin{aligned}\mathbb{Z} &= \text{the integers},\\ \mathbb{N} &= \text{the nonnegative integers},\\ [n] &= \{1, 2, \ldots, n\}.\end{aligned}$$

The *Stirling numbers of the 2nd kind* are defined for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ by $S(0, k) = \delta_{0,k}$ (Kronecker delta) and for $n \geq 1$

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

A *partition of S into k blocks* is $\rho = S_1/\ldots/S_k$ were we have $S = \uplus_i S_i$ and $S_i \neq \emptyset$ for all i. Let $S([n], k)$ be the set of ρ partitioning $[n]$ into k blocks.

Theorem

$$S(n, k) = \#S([n], k).$$

Ex. If $n = 3$ then

k	1	2	3
$S([3], k)$	123	1/23, 2/13, 3/12	1/2/3
$S(3, k)$	1	3	1

Let q be a variable and $n \in \mathbb{N}$. The usual *q-analogue of n* is

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

We may use $[n]$ for $[n]_q$ if no confusion will result. The *q-Stirling numbers of the 2nd kind* are $S[0,k] = \delta_{0,k}$ and for $n \geq 1$

$$S[n,k] = S[n-1,k-1] + [k]_q S[n-1,k].$$

The $S[n,k]$ were discovered by Carlitz (1948) and since studied by many authors (Garsia, Gould, Milne, S, Steingrímsson, Remmel, Wachs, White, Zeng, Zhang, etc.). The *type B Stirling numbers of the second kind* are $S_B(0,k) = \delta_{0,k}$ and for $n \geq 1$

$$S_B(n,k) = S_B(n-1,k-1) + (2k+1)S_B(n-1,k),$$

with q-analogue $S_B[n,k]$ obtained by replacing $2k+1$ by $[2k+1]_q$ in the $S_B(n,k)$ recursion. The case $q = 1$ is implicit of work of Dowling and Zaslavsky, and explicit in papers of Dolgachev-Lunts and Reiner. For general q, they only appear in a preprint of Swanson and Wallach. Some of our results have been independently found by Bagno, Garber, and Komatsu.

If $n \in \mathbb{N}$ then we will use the notation

$$\langle n \rangle = \{-n, -n+1, \ldots, n-1, n\}.$$

A *type B partition* of $\langle n \rangle$ is $\rho = S_0/S_1/S_2/\ldots/S_{2k}$ with

1. $0 \in S_0$ and if $i \in S_0$ then $-i \in S_0$, and
2. for $i \geq 1$ we have $S_{2i} = -S_{2i-1}$,

where $-S = \{-s \ : \ s \in S\}$. Call S_{2i} and S_{2i-1} *paired*. Let $S_B(\langle n \rangle, k)$ be the set of such ρ. Write $\overline{s}$ for $-s$.

Ex. An element of $S_B(\langle 5 \rangle, 2)$ is

$$\rho = 0\overline{1}1\overline{3}3 \ / \ \overline{4}/4 \ / \ 2\overline{5}/\overline{2}5.$$

Theorem

$$S_B(n, k) = \#S_B(\langle n \rangle, k).$$

Proof. Show that $\#S_B(\langle n \rangle, k)$ has the same recursion as $S_B(n, k)$. Given $\rho \in S_B(\langle n \rangle, k)$, let ρ' be ρ with $\pm n$ removed. If $\pm n$ are singletons in ρ then $\rho' \in S_B(\langle n-1 \rangle, k-1)$. Otherwise $\rho' \in S_B(\langle n-1 \rangle, k)$, and each such ρ' gives rise to $2k+1$ possible ρ since n can be inserted in any block of ρ'. $\square$

Let $|S| = \{|s| \ : \ s \in S\}$, so $|S_{2i}| = |S_{2i-1}|$ for $i \geq 1$. For all i let

$$m_i = \min |S_i|.$$

We will always write signed partitions in *standard form* where

1. $m_{2i} \in S_{2i}$ for all i, and
2. $0 = m_0 < m_2 < m_4 < \cdots < m_{2k}$.

Ex. The partition $\rho = 0\bar{1}1\bar{3}3 \ / \ \bar{4}/4 \ / \ 2\bar{5}/\bar{2}5$ has standard form

$$\rho = 0\bar{1}1\bar{3}3 \ / \ \bar{2}5/2\bar{5} \ / \ \bar{4}/4.$$

An *inversion* of ρ in standard form is a pair (s, S_j) satisfying

1. $s \in S_i$ for some $i < j$, and
2. $s > m_j$.

Let $\operatorname{inv} \rho$ be the number of inversions of ρ.

Ex. We have $\operatorname{inv}(0\bar{1}1\bar{3}3 \ / \ \bar{2}5/2\bar{5} \ / \ \bar{4}/4) = 5$ with inversions

$$(3, S_1), \ (3, S_2), \ (5, S_2), \ (5, S_3), \ (5, S_4).$$

Theorem (S-Swanson)

$$S_B[n,k] = \sum_{\rho \in S_B(\langle n \rangle, k)} q^{\operatorname{inv} \rho}.$$

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a set of variables. The *kth complete homogenenous symmetric polynomial in* $\mathbf{x}$ is

$$h_k(n) = \text{ sum of all monomials in } \mathbf{x} \text{ of degree } k.$$

Ex. $h_2(3) = x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2.$

Theorem

$$h_k(n) = h_k(n-1) + x_n h_{k-1}(n)$$

and

$$\sum_{k \geq 0} h_k(n) t^k = \prod_{i=1}^{n} \frac{1}{1 - x_i t}.$$

Corollary (S-Swanson)

$$S_B[n,k] = h_{n-k}([1],[3],\ldots,[2k+1])$$

and

$$\sum_{n \geq k} S_B[n,k] t^n = \frac{t^k}{(1-[1]t)(1-[3]t)\cdots(1-[2k+1]t)}.$$

Given a variable t and $k \in \mathbb{N}$ the corresponding *falling factorial* is

$$t\downarrow_k = t(t-1)(t-2)\cdots(t-k+1).$$

Ex. $t\downarrow_3 = t(t-1)(t-2).$

Theorem

$$t^n = \sum_{k=0}^{n} S(n,k)t\downarrow_k .$$

For variables $\mathbf{x}$ and t, the corresponding $\mathbf{x}$-*falling factorial* is

$$t\downarrow_k^{\mathbf{x}} = (t-x_1)(t-x_2)\cdots(t-x_k).$$

Ex. $t\downarrow_3^{\mathbf{x}} = (t-x_1)(t-x_2)(t-x_3).$

Theorem (S-Swanson)

$$t^n = \sum_{k=0}^{n} h_{n-k}(k+1)\ t\downarrow_k^{\mathbf{x}} .$$

and

$$t^n = \sum_{k=0}^{n} S_B[n,k](t-[1])(t-[3])\cdots(t-[2k-1]).$$

The *symmetric group*, $\mathfrak{S}_n$, is the group of permutations of $[n]$. It is the Coxeter group A_{n-1}. The *Stirling numbers of 1st kind* are

$$s(n,k) = (-1)^{n-k}(\# \text{ of } \pi \in \mathfrak{S}_n \text{ with } k \text{ cycles}).$$

Let $s([n],k)$ be the permutations counted by $s(n,k)$.

Ex. If $n = 3$ then

k	1	2	3
$s([3],k)$	$(1,2,3),\ (1,3,2)$	$(1)(2,3),\ (2)(1,3),\ (3)(1,2)$	$(1)(2)(3)$
$s(3,k)$	2	-3	1

Permutation $\pi = c_1 \cdots c_k \in s([n],k)$ has *underlying partition* $\rho = S_1/\ldots/S_k \in S([n],k)$ where, for all i,

$$S_i = \text{the set of elements in } c_i.$$

Ex. The permutations $(1,4,2)(3,5)$ and $(1,2,4)(3,5)$ both have underlying partition $124/35$.

Let $\langle n \rangle' = \langle n \rangle - \{0\}$. The *hyperoctahedral group*, $\mathfrak{H}_n$, is the group of all bijections $\pi : \langle n \rangle' \to \langle n \rangle'$ with, for all $i \in \langle n \rangle'$,

$$\pi(-i) = -\pi(i)$$

It is the Coxeter group B_n.

Ex. It suffices to specify $\pi(i)$ for $i > 0$. Say $\pi \in \mathfrak{H}_5$ satisfies

$$\pi(1) = \overline{3},\ \pi(2) = \overline{5},\ \pi(3) = 1,\ \pi(4) = \overline{4},\ \pi(5) = \overline{2}.$$

in cycle notation: $(1, \overline{3}, \overline{1}, 3)\ (2, \overline{5})\ (\overline{2}, 5)\ (4, \overline{4}) \in s_B(\langle 5 \rangle', 1)$, with underlying partition: $\rho = 0\overline{1}1\overline{3}3\overline{4}4\ /\ \overline{2}5/2\overline{5} \in S_B(\langle 5 \rangle, 1)$.

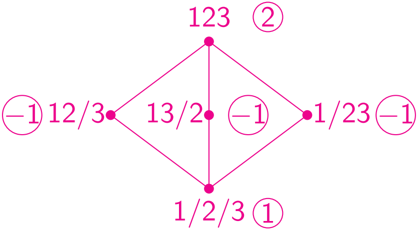
Every cycle c of $\pi \in \mathfrak{H}_n$ is of one of two types.

1. If $c = (a_1, a_2, \ldots, a_\ell)$ doesn't have both i and $-i$ for any i then π also contains *paired cycle* $-c = (-a_1, -a_2, \ldots, -a_\ell)$.
2. If c has both i and $-i$ for some i then c must have the form $c = (a_1, a_2, \ldots, a_\ell, -a_1, -a_2, \ldots, -a_\ell)$, an *unpaired cycle*.

Let $s_B(\langle n \rangle', k)$ be the set of all $\pi \in \mathfrak{H}_n$ with $2k$ paired cycles and $s_B(n, k) = (-1)^{n-k} \# s_B(\langle n \rangle', k)$. The *underlying partition* is defined as in type A with all unpaired cycles put in B_0 along with 0.

If $\rho = S_1/\dots/S_k$ and $\sigma = T_1/\dots/T_\ell$ are partitions of the same set then *ρ is a refinement of σ* if every S_i is contained in some T_j. Let Π_n and Π_n^B be the posets of partitions in $\uplus_k S([n],k)$ and $\uplus_k S_B(\langle n\rangle,k)$, respectively, ordered by refinement.

Ex. Π_3



Let P be a poset with a unique minimum element $\hat{0}$. The *Möbuis function of P* is $\mu : P \to \mathbb{Z}$ defined by $\mu(\hat{0}) = 1$ and for $x > \hat{0}$

$$\mu(x) = -\sum_{y<x} \mu(y).$$

Theorem (S-Swanson)

If $\rho = S_0/\dots/S_{2k} \in \Pi_n^B$ then

$$\mu(\rho) = (-1)^{n-k}(\#\text{ of } \pi \in \mathfrak{H}_n \text{ with underlying partition } \rho).$$

Exponential generating functions. It is well known that

$$\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k.$$

We given type B analogues and q-analogues of this formula. Let

$$[n]! = [1][2]\cdots[n],$$

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{[n]!}{[k]![n-k]!},$$

$$\exp_q(x) = \sum_{n\geq 0}\frac{x^n}{[n]!}.$$

Theorem (S-Swanson)

1. $\displaystyle\sum_{n\geq 0} S_B(n,k)\frac{x^n}{n!} = \frac{1}{2^k k!}e^x(e^{2x}-1)^k.$

2. $\displaystyle\sum_{n\geq 0} S[n,k]\frac{x^n}{[n]!} = \frac{1}{q^{\binom{k}{2}}[k]!}\sum_{i=0}^{k}(-1)^{k-i}q^{\binom{k-i}{2}}\left[\begin{array}{c} k \\ i \end{array}\right]\exp_q([i]x).$

Open Problem: Find $\sum_{n\geq 0} s_B[n,k]x^n/[n]!$.

Coinvariant algebras. The *coinvariant algebra* of $\mathfrak{S}_n$ is

$$\mathrm{R}_n = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle h_1(n), \ldots, h_n(n) \rangle}.$$

This algebra has *Artin basis*

$$\{x_1^{m_1} \cdots x_n^{m_n} \mid 0 \leq m_i < i \text{ for all } i \in [n]\}.$$

If $(\mathrm{R}_n)_d$ is the degree d graded piece of R_n then its Hilbert series is

$$\sum_{d \geq 0} \dim(\mathrm{R}_n)_d \; q^d = [n]!.$$

Zabrocki considered a super coinvariant algebra of $\mathfrak{S}_n$, SR_n, which has a 2nd set of anticommuting variables $\{\theta_1, \ldots, \theta_n\}$.

Conjecture (Zabrocki)

$$\sum_{d,f \geq 0} \dim(\mathrm{SR}_n)_{d,f} q^d t^f = \sum_{k \geq 0} [k]! S[n,k] t^{n-k}.$$

Swanson and Wallach made a similar conjecture in type B. We conjecture analogues of the Artin basis in both type A and B which, if correct, would prove both conjectures.

THANKS FOR LISTENING!