Patterns in Set Partitions

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1. The First Definition

Given sets S, T and a function $f : S \to T$, apply f element-wise to objects built from S

Ex. 1. If $p = a_1 a_2 \ldots a_m$ is a word over S, possibly with repetitions, then $f(p) = f(a_1)f(a_2)\ldots f(a_m)$.

2. If $B \subseteq S$ then $f(B) = \{f(a) \mid a \in B\}$.

Given two positive integers $m, n \in \mathbb{P}$, we will let $[m, n] = \{m, m+1, \ldots, n\}$ and $[n] = [1, n]$.

Now if $S \subseteq \mathbb{P}$ has cardinality $\#S = n$ then the *reduction map* is the unique order-preserving bijection $r_S : S \to [n]$.

Ex. If $S = \{2, 5, 7, 8\}$ then

$$r_S(2) = 1,\ r_S(5) = 2,\ r_S(7) = 3,\ r_S(8) = 4.$$

Ordinary pattern containment can be defined: Let $q \in \mathfrak{S}_m$, then $p = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$ *contains* q if there is a subword $p' = a_{i_1} a_{i_2} \ldots a_{i_m}$ of p such that

$$r_S(p') = q$$

where $S = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$.

A *partition π of a set S* is a family of nonempty sets $B_1, B_2, \ldots, B_k$ such that $\uplus_i B_i = S$. The B_i are called the *blocks of π* and the number of blocks is the *length of π*, $l(\pi)$. We write $\pi = B_1/B_2/\ldots/B_k \vdash S$, e.g., $137/28/456/9 \vdash [9]$. Let

$$\Pi_n = \{\pi \vdash [n]\} \quad \text{and} \quad \Pi = \biguplus_{n\ge1} \Pi_n.$$

We say *σ is contained in π*, $\sigma \subseteq \pi$, if each block of σ is contained in some block of π.

Ex. $28/3/46 \subseteq 137/28/456/9$ because $28 \subseteq 28$, $3 \subseteq 137$, and $46 \subseteq 456$.

Now *π contains the pattern σ*, $\pi \sqsupseteq \sigma$, if there is a partition $\pi' \subseteq \pi$ with

$$r_S(\pi') = \sigma$$

where $\pi' \vdash S$.

Ex. If $\sigma = 13/2$ then $\pi = 14/236/5$ contains six copies of σ: $14/2$, $14/3$, $26/4$, $26/5$, $36/4$, $36/5$.

We say *π avoids σ* if $\pi \not\sqsupseteq \sigma$ and let

$$\Pi_n(\sigma) = \{\pi \in \Pi_n \mid \pi \not\sqsupseteq \sigma\} \text{ and } \Pi(\sigma) = \biguplus_{n\ge1} \Pi_n(\sigma).$$

2. Enumeration

Sequence $(a_n)_{n\geq 0}$ has egf $F(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

Subset $J \subseteq \mathbb{P}$ has egf $F_J(x) = \sum_{j\in J} \frac{x^j}{j!}$.

Theorem 1 *Let*

$a^J_{n,k} = \#\{\pi \in \Pi_n \mid l(\pi) = k$ *and block sizes in* $J\}$.

Then

$$\sum_{n\geq 0} a^J_{n,k} \frac{x^n}{n!} = \frac{F_J(x)^k}{k!}.$$

Let

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \overline{\exp}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\exp_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!} \qquad \overline{\exp}_m(x) = \sum_{n=1}^{m} \frac{x^n}{n!}$$

and

$$\mathcal{C}_n F(x) = \text{ the coefficient of } x^n/n! \text{ in } F(x).$$

Theorem 2 (S) *We have*

$$\Pi(1/2/\ldots/m) = \{\pi \mid l(\pi) < m\}, \qquad (1)$$

$$\#\Pi_n(1/2/\ldots/m) = \mathcal{C}_n \exp_{m-1}(\overline{\exp}\ x). \qquad (2)$$

$$\Pi(12\ldots m) = \{\pi \mid \#B < m \ \ \forall B \in \pi\}, \qquad (3)$$

$$\#\Pi_n(12\ldots m) = \mathcal{C}_n \exp(\overline{\exp}_{m-1} x). \qquad (4)$$

Proof for $\sigma = 12\ldots m$. First, $\pi \sqsupseteq \sigma$ iff there is a block B of π with $\#B \geq m$ proving (3). Also

$$\#\Pi_n(12\ldots m) = \sum_{k\geq 0} a_{n,k}^J \quad \text{where} \quad J = [m-1].$$

So by Theorem 1

$$\sum_{n\geq 0} \#\Pi_n(12\ldots m)\,\frac{x^n}{n!} = \sum_{k\geq 0}\sum_{n\geq 0} a_{n,k}^{[m-1]}\,\frac{x^n}{n!}$$

$$= \sum_{k\geq 0} \frac{(\overline{\exp}_{m-1}x)^k}{k!} = \exp(\overline{\exp}_{m-1}x). \qquad \blacksquare$$

Call σ and *involution* if all $B \in \sigma$ have $\#B = 1$ or 2.

Corollary 3 (Klazar, S) *We have*

$$\Pi(1/2/3) = \{\pi \mid \pi \text{ has at most 2 blocks}\},$$

$$\#\Pi_n(1/2/3) = 2^{n-1}.$$

$$\Pi(123) = \{\pi \mid \pi \text{ is an involution}\},$$

$$\#\Pi_n(123) = \sum_{i\geq 0} \binom{n}{2i}(2i)!!$$

Partition π is *layered* if it is of the form

$$\pi = [1,i]/[i+1,j]/\ldots/[k+1,n].$$

Theorem 4 (S) *We have*

$$\begin{aligned}
\Pi(13/2) &= \{\pi \mid \pi \text{ is layered}\},\\
\#\Pi_n(13/2) &= 2^{n-1}.\\
\Pi(12/3) &= \{\pi = B_1/\ldots/B_k \mid \min B_i = i\ \forall i\\
&\qquad \text{and } \exists i \text{ with } [k+1,n] \subseteq B_i\},\\
\#\Pi_n(12/3) &= \binom{n}{2} + 1.
\end{aligned}$$

Given $\sigma = C_1/\ldots/C_l \vdash [m]$ we let

$\sigma' = C_1'/\ldots/C_l'$ where $C_i' = \{m+1-a \mid a \in C_i\}\ \forall i$.

Ex. $\sigma = 135/24/6$ implies $\sigma' = 642/53/1$.

Proposition 5 (S) *We have*

$$\begin{aligned}
\Pi(\sigma') &= \{\pi' \mid \pi \in \Pi(\sigma)\}, \text{ and}\\
\#\Pi_n(\sigma') &= \#\Pi_n(\sigma) \text{ for } n \geq 1.
\end{aligned}$$

Since $(12/3)' = 1/23$, this completes the enumeration of partitions avoiding a single pattern of length at most three. Sets of patterns and refinements by parity can also be enumerated.

3. The Stanley-Wilf Conjecture

Conjecture 6 (Stanley-Wilf) *For every permutation pattern q the limit*

$$\lim_{n\to\infty} \#\mathfrak{S}_n(q)^{1/n}$$

exists and is finite.

This is not always true of the limit

$$\lim_{n\to\infty} \#\Pi_n(\sigma)^{1/n}. \tag{5}$$

Theorem 7 (K) *If $\sigma = 123$ or $12/34$, then (5) is infinite.*

Proof for $\sigma = 123$. By Corollary 3 and Stirling

$$\#\Pi_n(123)^{1/n} \geq n!!^{1/n} \geq C\sqrt{n}$$

for some constant C and large n. ■

Since $\tau \sqsupseteq \sigma$ implies $\Pi_n(\tau) \supseteq \Pi_n(\sigma)$, we have

Corollary 8 (K) *The limit (5) is infinite if $\sigma \sqsupseteq 123$ or $\sigma \sqsupseteq 12/34$.*

So the patterns for which we have not yet determined (5) are exactly those in $\Pi(123, 12/34)$

Another description of the $\sigma \in \Pi(123, 12/34)$: Since σ avoids 123, it is an involution. To include the other restriction, given a permutation $p = a_1 a_2 \ldots a_n$ we have a corresponding *permutation partition*

$$\sigma_p = 1(a_1 + n)/2(a_2 + n)/\ldots/n(a_n + n).$$

Ex. If $p = 2431$ then $\sigma_p = 16/28/37/45$.

An *inflated permutation partition* is $\sigma = \tau \uplus \rho$ such that τ consists of only singleton blocks and $r_S(\rho)$ is a permutation partition where $\rho \vdash S$.

Theorem 9 (K) *1. The inflated permutation partitions are precisely those in $\Pi(123, 12/34)$.*

2. If for each permutation partition there is a constant c with $\Pi_n(\sigma_p) \leq c^n$ for all large n, then the same is true of inflated permutation partitions.

Theorem 10 (S) *The limit (5) exists and is finite for $\sigma = 1/2/\ldots/m$ and for all partitions with at most 3 elements except $\sigma = 123$.*

A limit either exists and is finite, exists and is infinite, or does not exist. Also $\#\Pi_n(\sigma) \leq c^n$ for all large n implies the limit is finite or does not exist.

Define $\sigma \vdash [m]$ to be *reducible* if there is j with $0 < j < m$ such that $\sigma = \tau \uplus \rho$ with

$$\tau \vdash [j] \quad \text{and} \quad \rho \vdash [j+1, m].$$

Using Fekete's Lemma one can prove

Theorem 11 (S) *For σ irreducible, the limit (5) exisits (and may be infinite).*

Since permutation patterns are irreducible, this Theorem implies Arratia's result that the corresponding limit exists for all permutations.

Putting all the evidence together, we conjecture

Conjecture 12 (K,S) *For all σ, the limit (5) exists. It is finite precisely for $\sigma \in \Pi_n(123, 12/34)$.*

Notes: 1. Since $p \in \mathfrak{S}_n(q)$ implies $\sigma_p \in \Pi_n(\sigma_q)$ we have $\#\Pi_n(\sigma_q) \geq \#\mathfrak{S}_n(q)$. So the previous conjecture implies the Stanley-Wilf conjecture.

2. Klazar has shown that for $\sigma \in \Pi_n(123, 12/34)$ we have $\#\Pi_n(\sigma) \leq \omega(n)^n$ where $\omega(n)$ grows slowly.

4. The Noonan-Zeilberger Conjecture

A sequence $(a_n)_{n\geq 0}$ is *P-recursive* if there are polynomials $P_0(n), P_1(n), \ldots, P_k(n)$ (not all zero) such that for all $n \in \mathbb{P}$

$$P_0(n)a_n + P_1(n)a_{n+1} + \cdots + P_k(n)a_{n+k} = 0.$$

Ex. $a_n = n!$ is P-recursive: $(n+1)a_n - a_{n+1} = 0$.

Conjecture 13 (Noonan-Zeilberger) *For any permutation q, the sequence $a_n = \#\mathfrak{S}_n(q)$ is P-recursive.*

A series $f(x) \in \mathbb{C}[[x]]$ is *D-finite* if there are polynomials $p_0(x), p_1(x), \ldots, p_k(x)$ (not all zero) such that

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_k(x)f^{(k)}(x) = 0.$$

Sequence $(a_n)_{n\geq 0}$ has ogf $f(x) = \sum_{n\geq 0} a_n x^n$.

Theorem 14 (Jungen,Stanley) *1. $(a_n)_{n\geq 0}$ is P-recursive iff its ogf $f(x)$ is D-finite.*

2. If $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are P-recursive then so is $(a_n b_n)_{n\geq 0}$.

Corollary 15 (S) *$(a_n)_{n\geq 0}$ is P-recursive iff its egf $F(x)$ is D-finite.*

A series $f(x) \in \mathbb{C}[[x]]$ is *algebraic* if there are polynomials $q_0(x), q_1(x), \ldots, q_k(x)$ (not all zero) such that

$$q_0(x)f(x) + q_1(x)f(x)^2 + \cdots + q_k(x)f(x)^k = 0.$$

Theorem 16 (Stanley) *If $f(x)$ is D-finite and $g(x)$ is algebraic then $f(g(x))$ is D-finite.*

Theorem 17 (S) *If σ is a pattern with at most 3 elements or $\sigma = 1/2/\ldots/m$ or $\sigma = 12\ldots m$ then the sequence $a_n = \#\Pi_n(\sigma)$ is P-recursive.*

Proof for $\sigma = 12\ldots m$. We showed that

$$\sum_{n\geq 0} \#\Pi_n(12\ldots m)\frac{x^n}{n!} = \exp(\overline{\exp}_{m-1}x).$$

Now $\overline{\exp}_{m-1}(x)$ is a polynomial and so algebraic. Furthermore, $f(x) = \exp(x)$ is D-finite since we have $f(x) - f'(x) = 0$. So we are done by the previous Theorem and Corollary. ■

Conjecture 18 (S) *For any partition σ, we have that the sequence $a_n = \#\Pi_n(\sigma)$ is P-recursive.*

5. The Second Definition

A sequence $r = a_1 a_2 \ldots a_n$ of positive integers is a *restricted growth function (RGF) of length n* if

$$a_1 = 1 \text{ and } a_i \le 1 + \max_{j<i} a_j \text{ for } i \ge 2.$$

Let R_n be the set of such functions and $R = \uplus_{n\ge 1} R_n$. To connect with partitions, all $\pi = B_1/B_2/\ldots/B_k$ in this section will be *ordered*, i.e., indexed so that

$$\min B_1 < \min B_2 < \ldots < \min B_k. \tag{6}$$

Given $\pi \vdash [n]$ we construct a sequence $r_\pi = a_1 a_2 \ldots a_n$ by letting $a_i = j$ iff $i \in B_j$.

Ex. If $\pi = 137/28/456/9$ then $r_\pi = 121333124$.

Condition (6) forces r_π to be an RGF and this map is a bijection $\Pi_n \longleftrightarrow R_n$.

Now $r \in R$ *contains the pattern s*, $r \succeq s$, if there is a subword $r' = a_{i_1} a_{i_2} \ldots a_{i_m}$ of r such that

$$r_S(r') = s$$

where $S = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$.

In terms of partitions, $\pi \succeq \sigma$ if the elements reducing to the jth block of σ come from B_{i_j} in π with

$$i_1 < i_2 < \ldots < i_l.$$

Ex. If $\sigma = 13/2$ then $\pi = 14/236/5 \succeq \sigma$ in four ways: 14/2, 14/3, 26/5, 36/5.

We say π *RGF-avoids* σ if $\pi \not\succeq \sigma$ and let

$$R_n(\sigma) = \{\pi \in \Pi_n \mid \pi \not\succeq \sigma\} \text{ and } R(\sigma) = \biguplus_{n \geq 1} R_n(\sigma).$$

Note that $\pi \succeq \sigma$ implies $\pi \sqsupseteq \sigma$ so $R_n(\sigma) \supseteq \Pi_n(\sigma)$.

Many of the results for $\sqsupseteq$ have analogues for $\succeq$.

Theorem 19 (S) *If* $\sigma \vdash [3]$ *then*

$$\#R_n(\sigma) = 2^{n-1}$$

except for $\sigma = 123$ *when*

$$R_n(123) = \Pi_n(123).$$

Conjecture 20 (S) *The analogues of the Stanley-Wilf and Noonan-Zeilberger conjectures obtained by replacing* $\sqsupseteq$ *with* $\succeq$ *everywhere are also true.*

6. More Open Problems

a. Associated with any ordered partition π we have a permutation p_π obtained by concatenating the blocks where each block is written in decreasing order.

Ex. If $\pi = 14/236/5$ then $p_\pi = 416325$.

Note that if π is layered then so is p_π and any layered permutation is p_π for a unique π. Also, if σ, π are layered then $\sigma \sqsubseteq \pi$ iff $\sigma \succeq \pi$. These observations lead to the following analogue of a result of Stromquist.

Theorem 21 (S) *Let σ be layered. Among all elements of Π_n which contain the maximum number of copies of σ (using either $\sqsubseteq$ or $\succeq$) there is always one which is layered.*

What can be said about packing densities of partition patterns, especially layered ones?

b. One can define *generalized RGF-pattern containment* by insisting that certain adjacent elements of the pattern RGF must be adjacent in the containing RGF à la Babson and Steingrímsson. What can be said about the concepts we have addressed in this context?

c. A poset is *well partially ordered (WPO)* if it contains neither an infinite descending chain nor an infinite antichain. We have the following weak analogue of a theorem of Atkinson, Murphy, and Ruškuc.

Theorem 22 *(S) The poset* $(\Pi(\sigma), \preceq)$ *is WPO if* σ *has at most three elements with the exception of* $(\Pi(123), \preceq)$ *which is not WPO.*

AMR actually give a characterization of the permutations q such that $\mathfrak{S}(q)$ is WPO. It would be interesting to do the same for partitions.