Permutation Patterns and Statistics

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joint work with

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Pattern containment and avoidance

Permutation statistics: inversions

Permutation statistics: major index

$q$-Catalan numbers

Multiple restrictions

Future work
Outline

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Two sequences of distinct integers $\pi = a_1 a_2 \ldots a_k$ and $\sigma = b_1 b_2 \ldots b_k$ are order isomorphic if, for all $i$ and $j$,

$$a_i < a_j \iff b_i < b_j.$$
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Ex. The sequences $\pi = 132$ and $\sigma = 485$ are order isomorphic.
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Ex. The sequences $\pi = 132$ and $\sigma = 485$ are order isomorphic. Let $\mathcal{S}_n$ be the symmetric group of all permutations of $\{1, \ldots, n\}$ and let $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$. 

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Theorem

For any $\pi \in \mathcal{S}_3$ we have $\# \text{Av}_n(\pi) = C_n$, the nth Catalan number.
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Let \( \mathcal{S}_n \) be the **symmetric group** of all permutations of \( \{1, \ldots, n\} \) and let \( \mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n \). If \( \pi, \sigma \in \mathcal{S} \) the \( \sigma \) **contains \( \pi \) as a pattern** if there is a subsequence \( \sigma' \) of \( \sigma \) order isomorphic to \( \pi \).

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We say \( \sigma \) **avoids \( \pi \)** if \( \sigma \) does not contain \( \pi \) and let

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Let $S_n$ be the symmetric group of all permutations of $\{1, \ldots, n\}$ and let $S = \bigcup_{n \geq 0} S_n$. If $\pi, \sigma \in S$ the $\sigma$ contains $\pi$ as a pattern if there is a subsequence $\sigma'$ of $\sigma$ order isomorphic to $\pi$.

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The *diagram* of $\pi = a_1 \ldots a_n$ is $(1, a_1), \ldots, (n, a_n) \in \mathbb{Z}^2$. 

\[132 = R_{90}(132) = 231\]

The dihedral group $D_4$ of symmetries of the square acts on $S_n$:

$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_0, r_1, r_{-1}, r_{\infty}\}$

where $R_{\theta}$ is rotation counter-clockwise through $\theta$ degrees and $r_m$ is reflection in a line of slope $m$. Note that for any $\rho \in D_4$:

$\sigma$ contains $\pi$ $\iff$ $\rho(\sigma)$ contains $\rho(\pi)$,

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These Wilf equivalences are called *trivial*. 
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\includegraphics[width=1.5in]{square_diagram132.png}
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$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv} \sigma} = 1(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \overset{\text{def}}{=} [n]_q!.$$
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Call \( \pi \) and \( \pi' \) inv-Wilf equivalent, \( \pi \overset{\text{inv}}{=} \pi' \), if \( l_n(\pi; q) = l_n(\pi'; q) \) for all \( n \geq 0 \).
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Note that \((i, j)\) is an inversion of \(\pi\) iff the line connecting the corresponding points in the diagram of \(\pi\) has negative slope.
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**Proposition (DDJSS)**

*Let \(\pi \in \mathcal{S}\) and \(\rho \in D_4\). Then*

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\operatorname{inv} \rho(\pi) = \operatorname{inv} \pi \iff \rho \in \{R_0, R_{180}, r_1, r_{-1}\}.
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The inv-Wilf equivalences in this proposition are call *trivial*. 
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Let \([\pi]_{\text{inv}}\) denote the inv-Wilf equivalence class of \(\pi\).
Theorem (DDJSS)

*The inv-Wilf equivalence classes for \( \pi \in \mathfrak{S}_3 \) are*

\[
[123]_{\text{inv}} = \{123\}, \\
[321]_{\text{inv}} = \{321\}, \\
[132]_{\text{inv}} = \{132, 213\}, \\
[231]_{\text{inv}} = \{231, 312\}.
\]

*Proof.*

The two equivalences follow from the proposition:

\[ 213 = R_{180}(132) \text{ and } 312 = R_{180}(231). \]

To see that there are no others, note that for \( \pi \in \mathfrak{S}_k \)

\[
I_k(\pi; q) = \sum_{\sigma \in \mathfrak{S}_k \setminus \{\pi\}} q^{\text{inv}\, \sigma} = [k] \cdot q^k - q^{\text{inv}\, \pi}.
\]

So if \( \pi, \pi' \in \mathfrak{S}_k \) with \( \pi_{\text{inv}} \equiv \pi'_{\text{inv}} \) then \( \text{inv}\, \pi = \text{inv}\, \pi' \).

Finally, check that any 2 classes above have differing inversion numbers.

Conjecture

All inv-Wilf equivalences are trivial.
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So if $\pi, \pi' \in S_k$ with $\pi \equiv^\text{inv} \pi'$ then $\text{inv}\pi = \text{inv}\pi'$. 
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So if $\pi, \pi' \in S_k$ with $\pi \equiv \pi'$ then $\text{inv } \pi = \text{inv } \pi'$. Finally, check that any 2 classes above have differing inversion numbers. \qed
Theorem (DDJSS)

The inv-Wilf equivalence classes for $\pi \in \mathfrak{S}_3$ are

\[
[123]_{\text{inv}} = \{123\}, \\
[321]_{\text{inv}} = \{321\}, \\
[132]_{\text{inv}} = \{132, 213\}, \\
[231]_{\text{inv}} = \{231, 312\}.
\]

Proof. The two equivalences follow from the proposition:

\[213 = R_{180}(132) \quad \text{and} \quad 312 = R_{180}(231)\]

To see that there are no others, note that for $\pi \in \mathfrak{S}_k$

\[I_k(\pi; q) = \sum_{\sigma \in \mathfrak{S}_k - \{\pi\}} q^{\text{inv}\sigma} = [k]q! - q^{\text{inv}\pi}.\]

So if $\pi, \pi' \in \mathfrak{S}_k$ with $\pi \equiv^\text{inv} \pi'$ then $\text{inv} \pi = \text{inv} \pi'$. Finally, check that any 2 classes above have differing inversion numbers. □

Conjecture

All inv-Wilf equivalences are trivial.
Outline

Pattern containment and avoidance

Permutation statistics: inversions

Permutation statistics: major index

$q$-Catalan numbers

Multiple restrictions

Future work
The *major index* of $\pi = a_1 \ldots a_n$ is

$$\text{maj } \pi = \sum_{a_i > a_{i+1}} i.$$
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$$\text{maj} \, \pi = \sum_{a_i > a_{i+1}} i.$$ 

Ex. If $\pi = 253614$ then $\text{maj} \, \pi = 2 + 4 = 6$. 

Theorem (MacMahon)

$$\sum_{\sigma \in S_n} q^{\text{maj} \, \sigma} = [n] \, q!.$$ 

Given $\pi \in S_n$ we have a corresponding major index polynomial

$$M_n(\pi; q) = \sum_{\sigma \in \text{Av}_n(\pi)} q^{\text{maj} \, \sigma}.$$ 

Call $\pi, \pi' \, \text{maj-Wilf equivalent}$, $\pi \, \text{maj} \equiv \pi'$, if $M_n(\pi; q) = M_n(\pi'; q)$ for all $n \geq 0$. 

Let $[\pi] \, \text{maj}$ denote the maj-Wilf equivalence class of $\pi$. 

Note: No $\rho \in D_4$ preserves the major index.
The major index of $\pi = a_1 \ldots a_n$ is

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Theorem (MacMahon)

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = [n]_q!.$$  

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The maj-Wilf equivalence classes for $\pi \in S_3$ are

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\end{align*}
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If \( \pi = a_1 \ldots a_n \) and \( \sigma_1, \ldots, \sigma_n \in S \) then the inflation of \( \pi \) by the \( \sigma_i \) is the permutation \( \pi[\sigma_1, \ldots, \sigma_n] \) whose diagram is obtained from that of \( \pi \) by replacing the \( i \)th dot with a copy of \( \sigma_i \) for all \( i \).
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Ex.

\[
132 = \begin{array}{c}
\bullet \\
\end{array} \quad 132[\sigma_1, \sigma_2, \sigma_3] = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \\
\end{array}
\]

Conjecture

For all $m, n \geq 0$ we have:

\[
132[\iota_m, 1, \delta_n] \equiv_{\text{maj}} 231[\iota_m, 1, \delta_n],
\]

where $\iota_m = 12 \ldots m$ and $\delta_n = n(n-1) \ldots 1$. 
Theorem (DDJSS)

The maj-Wilf equivalence classes for $\pi \in S_3$ are

$$[123]_{\text{maj}} = \{123\},$$
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\bullet
\end{array}
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\begin{array}{c}
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\end{array}
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Since $\# \text{Av}_n(\pi) = C_n$ for any $\pi \in S_3$, the corresponding $I_n(\pi; q)$ and $M_n(\pi; q)$ are $q$-analogues of the Catalan numbers since setting $q = 1$ we recover $C_n$. 
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$$C_n(q) = I_n(132; q) = I_n(213; q)$$
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were introduced by Carlitz and Riordan and studied by numerous authors but the others seem to be new.
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C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}.
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**Theorem (DDJSS)**

For $n \geq 1$:

$$I_{n}(312; q) = \sum_{k=0}^{n-1} q^k I_{k}(312; q) I_{n-k-1}(312; q).$$
Since \( \# \text{Av}_n(\pi) = C_n \) for any \( \pi \in S_3 \), the corresponding \( l_n(\pi; q) \) and \( M_n(\pi; q) \) are \( q \)-analogues of the Catalan numbers since setting \( q = 1 \) we recover \( C_n \). The polynomials
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\]

**Theorem (DDJSS)**

For \( n \geq 1 \):
\[
l_n(312; q) = \sum_{k=0}^{n-1} q^k l_k(312; q) l_{n-k-1}(312; q).
\]

**Conjecture**

For \( n \geq 1 \):
\[
l_n(321; q) = l_{n-1}(321; q) + \sum_{k=0}^{n-2} q^{k+1} l_k(321; q) l_{n-k-1}(321; q).
\]
Divisibility properties of Catalan numbers has been a topic of recent interest: Deutsch & Sagan; Eu, Liu, & Yeh; Kauers, Krattenthaler & Müller; Konvalinka; Lin; Liu & Yeh; Postnikov & Sagan; Xin & Xu; Yildiz.
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**Theorem**

*We have that $C_n$ is odd if and only if $n = 2^k - 1$ for some $k \geq 0$.*
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For any polynomial $f(q)$ we let

$$\langle q^i \rangle f(q) = \text{the coefficient of } q^i \text{ in } f(q).$$
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**Theorem (DDJSS)**

_For all \( k \geq 0 \) we have

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\langle q^i \rangle l_{2^k-1}(321; q) = \begin{cases} 
1 & \text{if } i = 0, \\
an \text{ even number} & \text{if } i \geq 1.
\end{cases}
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Future work
If $\Pi \subseteq \mathcal{S}$ then we let

$$\text{Av}_n(\Pi) = \{ \sigma \in \mathcal{S}_n : \sigma \text{ avoids } \pi \text{ for all } \pi \in \Pi \}.$$
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Simion & Schmidt classified $\# \text{Av}_n(\Pi)$ for all $\Pi \subseteq \mathcal{S}_3$ including:

- $\# \text{Av}_n(132, 231) = 2^{n-1}$
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We have classified $I_n(\Pi; q)$ and $M_n(\Pi; q)$ for $\Pi \subseteq \mathcal{S}_3$.

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We have

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We have

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For some polynomials we could not give explicit expressions and so instead gave recursions or generating functions.
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\[ M(\Pi; q, x) = \sum_{n \geq 0} M_n(\Pi; q)x^n, \]
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**Theorem (DDJSS)**

\[ M(231, 321; q, x) = \sum_{k \geq 0} \frac{q^{k^2}x^{2k}}{(x)_k(x)_{k+1}}. \]
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**Proof sketch.** If \( \sigma = a_1 \ldots a_n \in \text{Av}_n(231, 321) \) then \( \sigma \) is determined by its left-right maxima (lrn).
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For some polynomials we could not give explicit expressions and so instead gave recursions or generating functions. Define

\[ M(\Pi; q, x) = \sum_{n \geq 0} M_n(\Pi; q)x^n, \]

and

\[ (x)_k = (1 - x)(1 - qx)(1 - q^2x) \ldots (1 - q^{k-1}x). \]

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Outline

Pattern containment and avoidance

Permutation statistics: inversions

Permutation statistics: major index

$q$-Catalan numbers

Multiple restrictions

Future work
1. What happens if one considers permutations in $\mathfrak{S}_n$ for $n \geq 3$?
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4. What happens if one looks at pattern avoidance in other combinatorial structures such as words or set partitions?
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