# Permutation Patterns and Statistics

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

joint work with

T. Dokos (Ohio State), T. Dwyer (U. Florida), B. Johnson (Michigan State), and K. Selsor (U. South Carolina)

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Pattern containment and avoidance

Permutation statistics: inversions

Permutation statistics: major index

$q$-Catalan numbers

Multiple restrictions

Future work

# Outline

Pattern containment and avoidance

Permutation statistics: inversions

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Two sequences of distinct integers $\pi = a_1 a_2 \dots a_k$ and $\sigma = b_1 b_2 \dots b_k$ are *order isomorphic* if, for all $i$ and $j$,

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**Theorem**

*For any $\pi \in \mathfrak{S}_3$ we have $\# \mathrm{Av}_n(\pi) = C_n$, the nth Catalan number.*

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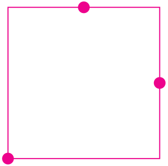
$132 =$ 

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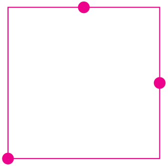
$132 =$ 

The dihedral group $D_4$ of symmetries of the square acts on $\mathfrak{S}_n$:

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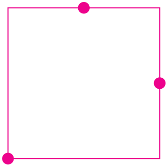
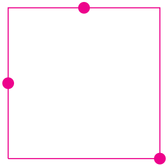
$132 =$  $R_{90}(132) =$  $= 231$

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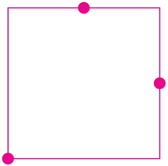
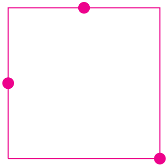
$132 =$  $R_{90}(132) =$  $= 231$

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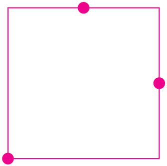
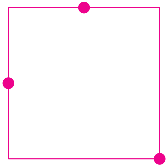
$132 =$  $R_{90}(132) =$  $= 231$

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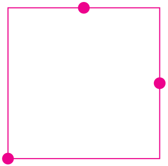
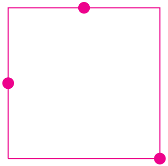
$132 =$  $R_{90}(132) =$  $= 231$

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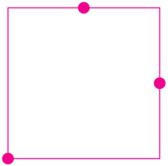
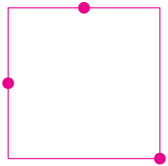
$132 =$  $R_{90}(132) =$  $= 231$

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These Wilf equivalences are called *trivial*.

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$$\#\,\mathrm{Av}_n(\pi) = I_n(\pi; 1) = I_n(\pi'; 1) = \# Av_n(\pi').$$

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Let $[\pi]_{\text{inv}}$ denote the inv-Wilf equivalence class of $\pi$.

## Theorem (DDJSS)

*The inv-Wilf equivalence classes for $\pi \in \mathfrak{S}_3$ are*

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**Proof.** The two equivalences follow from the proposition:

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## Conjecture

*All inv-Wilf equivalences are trivial.*

# Outline

Pattern containment and avoidance

Permutation statistics: inversions

Permutation statistics: major index

$q$-Catalan numbers

Multiple restrictions

Future work

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Note: No $\rho \in D_4$ preserves the major index.

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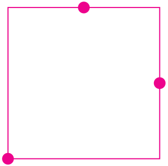
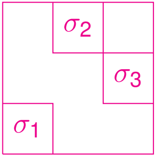
$132 =$ 

$132[\sigma_1, \sigma_2, \sigma_3] =$ 

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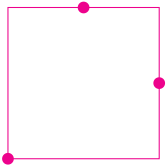
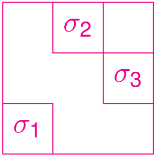
$132 =$ 

$132[\sigma_1, \sigma_2, \sigma_3] =$ 


## Conjecture

*For all $m, n \geq 0$ we have:* $132[\iota_m, 1, \delta_n] \overset{\text{maj}}{\equiv} 231[\iota_m, 1, \delta_n]$, *where $\iota_m = 12 \ldots m$ and $\delta_n = n(n-1) \ldots 1$.*

# Outline

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## Theorem (DDJSS)

*For $n \geq 1$:* $$I_n(312; q) = \sum_{k=0}^{n-1} q^k I_k(312; q) I_{n-k-1}(312; q).$$

Divisibility properties of Catalan numbers has been a topic of recent interest: Deutsch & Sagan; Eu, Liu, & Yeh; Kauers, Krattenthaler & Müller; Konvalinka; Lin; Liu & Yeh; Postnikov & Sagan; Xin & Xu; Yildiz.

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# Outline

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3. What happens if one uses generalized pattern avoidance where copies of a pattern are required to have certain pairs of elements in the diagram adjacent either horizontally or vertically?

1. What happens if one considers permutations in $\mathfrak{S}_n$ for $n \geq 3$?
2. What happens if one uses other statistics in place of inv and maj? Elizalde has studied the excedance and number of fixed points statistics.
3. What happens if one uses generalized pattern avoidance where copies of a pattern are required to have certain pairs of elements in the diagram adjacent either horizontally or vertically?
4. What happens if one looks at pattern avoidance in other combinatorial structures such as compositions or set partitions?

THANKS FOR LISTENING!