Probabilistic Proofs of Hooklength Formulas

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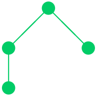
$T =$ 

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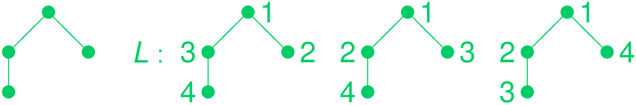

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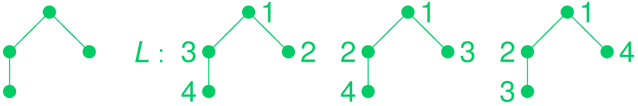

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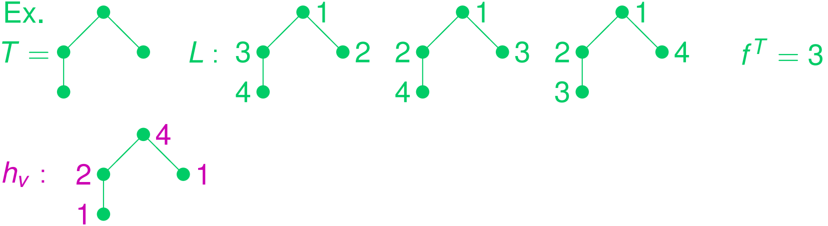

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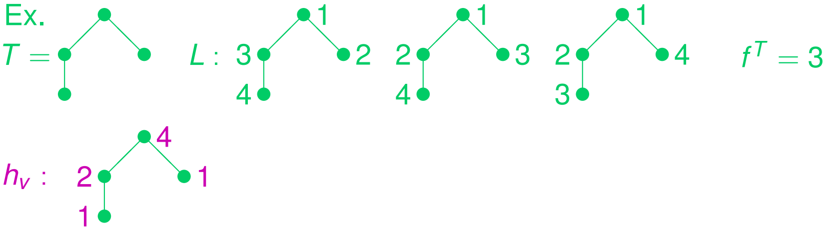

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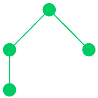

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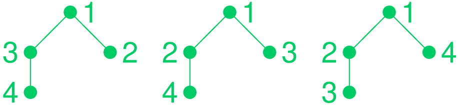

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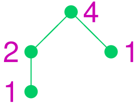

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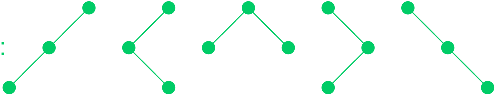

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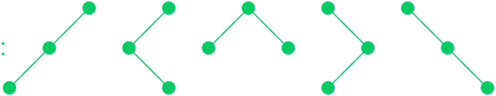

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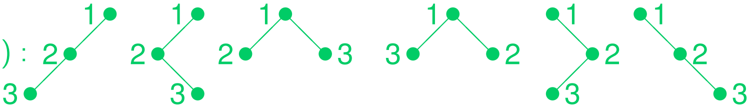

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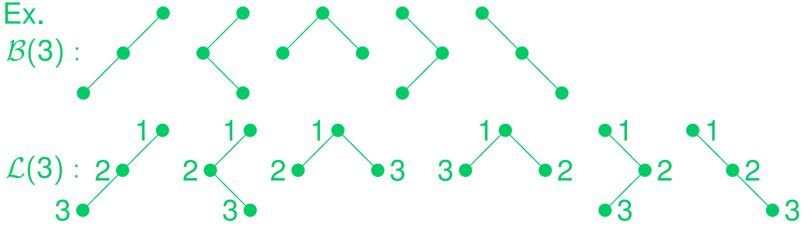

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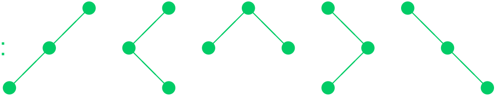

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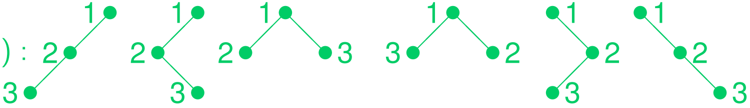

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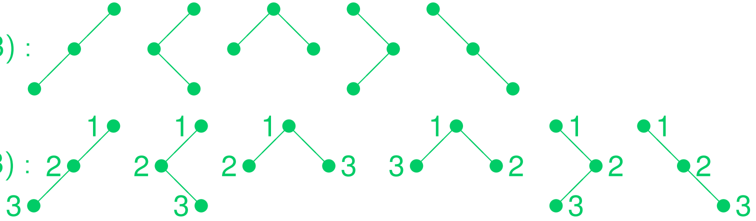

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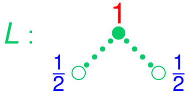

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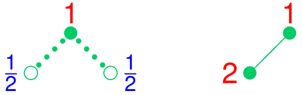

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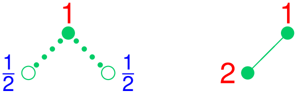

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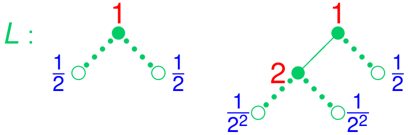

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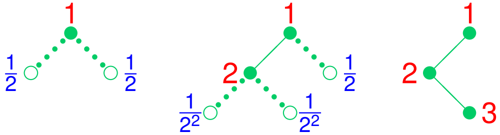

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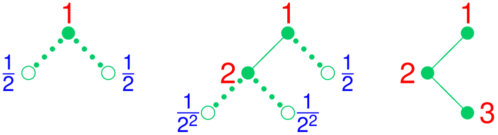

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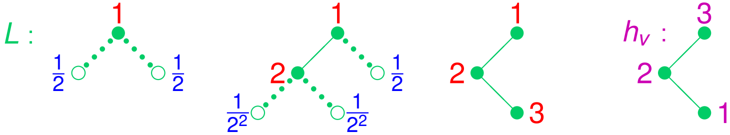

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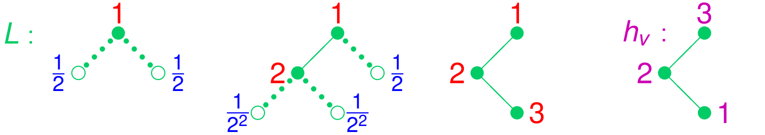

$$\text{prob } L = 1 \quad \cdot \quad \frac{1}{2} \quad \cdot \quad \frac{1}{2^2} \quad = \quad \prod_{v \in T} \frac{1}{2^{h_v - 1}}.$$

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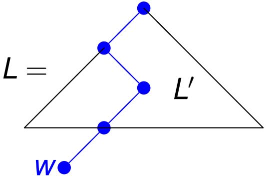

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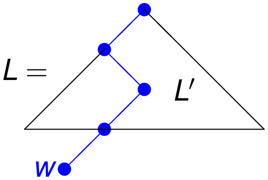

The hooklengths in L and L' are related by

$$h_v = \begin{cases} h'_v + 1 & \text{if } v \text{ is on the unique root-to-}w\text{ path } P, \\ h'_v & \text{else.} \end{cases}$$

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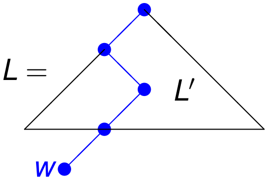

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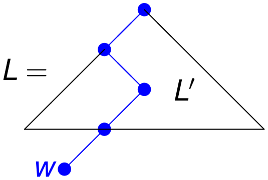

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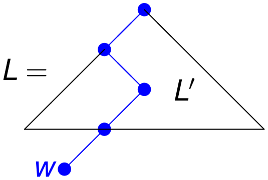

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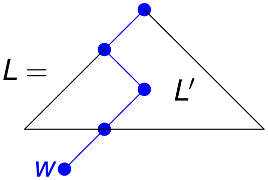

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Outline

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For any n

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For any n,

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(e) What is the analogue for tableaux of Han's formulas?