Introduction to Möbius Functions

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Motivation and Introduction

I. Number Theory. The number-theoretic Möbius function is $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free,} \\ (-1)^k & \text{if } n = \text{product of } k \text{ distinct primes,} \end{cases}$$

The importance of μ lies in the number-theoretic Möbius Inversion Theorem.

Theorem 1 *Let $f, g : \mathbb{Z}_{>0} \to \mathbb{Z}$ satisfy*

$$f(n) = \sum_{d|n} g(d)$$

for all $n \in \mathbb{Z}_{>0}$. Then

$$g(n) = \sum_{d|n} \mu(n/d) f(d). \qquad \blacksquare$$

II. Combinatorics. A very useful tool is the Principle of Inclusion-Exclusion or PIE.

Theorem 2 *Let S be a finite set and $S_1, \ldots, S_n \subseteq S$.*

$$|S - \bigcup_{i=1}^{n} S_i| \;=\; |S| - \sum_{1 \le i \le n} |S_i| + \sum_{1 \le i < j \le n} |S_i \cap S_j| - \cdots + (-1)^n |\bigcap_{i=1}^{n} S_i|. \qquad \blacksquare$$

III. Theory of Finite Differences. If one takes a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ then there is an analog of the derivative, namely the difference operator

$$\Delta f(n) = f(n) - f(n-1)$$

(where $f(-1) = 0$ by definition), and an analog of the integral, namely the summation operator

$$Sf(n) = \sum_{i=0}^{n} f(n).$$

The Fundamental Theorem of the Difference Calculus then states

Theorem 3 (FTDC) *If* $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ *then*

$$\Delta S f(n) = f(n). \qquad \blacksquare$$

One of the advantages of the combinatorial Möbius function is that its inversion theorem unifies and generalizes the previous three results. In addition, it makes the number-theoretic definition transparant, encodes topological information about poets, and has even been used to bound the running time of certain algorithms.

Möbius functions of posets

Let P be a finite *poset* (partially ordered set) which is a set P with an order relation $\leq$ satisfying the three axioms for all $x, y, z \in P$:

1. (reflexsive) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$,

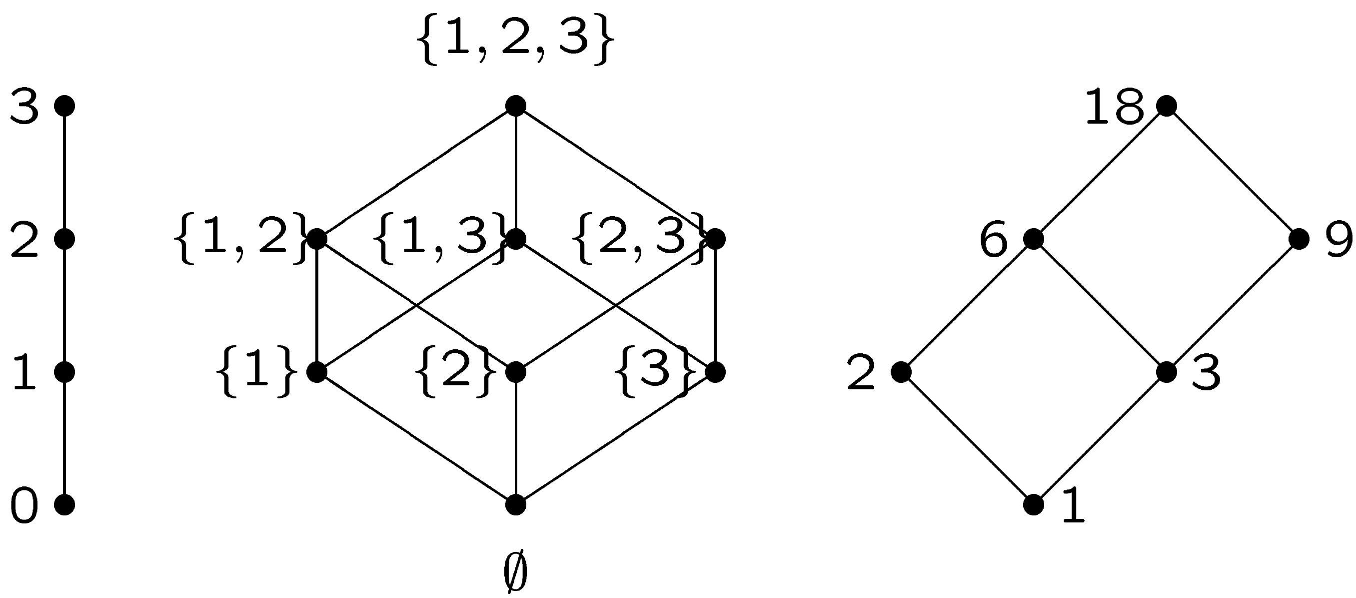
A poset is represented by its *Hasse diagram* which is a graph whose vertices are the elements of P and with an edge from x up to y if x is *covered* by y (i.e., $x < y$ and there is no z with $x < z < y$).

Examples:

1. The *chain*, C_n, consists of the integers $\{0, 1, \ldots, n\}$ ordered in the usual manner.

2. The *Boolean algebra*, B_n, has as elements all subsets of $[n] := \{1, 2, \ldots, n\}$ and $\subseteq$ as order relation.

3. The *divisor poset*, D_n, consists of all $d|n$ ordered by $c \leq d$ if $c|d$.



C_3 B_3 D_{18}

If P has a unique minimal element then it will be denoted $\hat{0} = \hat{0}_P$, and if it has a unique maximal element then we will use the notation $\hat{1} = \hat{1}_P$.

If P has a $\hat{0}$ then its *Möbius function*, $\mu : P \to \mathbf{Z}$, is defined recursively by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{if } x > \hat{0}. \end{cases}$$

Equivalently

$$\sum_{y \le x} \mu(y) = \delta_{x,\hat{0}}$$

where $\delta_{x,\hat{0}}$ is the Kronecker delta.

Computations: It is immediate directly from the definition of μ that in C_n we have

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ -1 & \text{if } x = 1 \\ 0 & \text{if } x \ge 2 \end{cases}$$

1. If $x \in B_n$ then $\mu(x) = (-1)^{|x|}$.

2. If $x \in D_n$ then $\mu(x)$ is the number theory function.

If P, Q are posets then order $P \times Q$ componentwise.

Proposition 4 *If $P \times Q$ has Möbius function μ then*

$$\mu(x, y) = \mu_P(x)\mu_Q(y)$$

Proof. It suffices to show that $\mu'(x, y) := \mu_P(x)\mu_Q(y)$ satisfies the defining equation for μ.

$$\begin{aligned} \sum_{(a,b)\leq(x,y)} \mu'(a, b) &= \sum_{(a,b)\leq(x,y)} \mu_P(a)\mu_Q(b) \\ &= \sum_{a\leq x} \mu_P(a) \sum_{b\leq y} \mu_Q(b) \\ &= \delta_{x,\hat{0}_P}\delta_{y,\hat{0}_Q} = \delta_{(x,y),(\hat{0}_P,\hat{0}_Q)}. \end{aligned} \quad \blacksquare$$

P and Q are *isomorphic*, $P \cong Q$, if there is a bijection $f : P \to Q$ with both f and f^{-1} order-preserving.

Proposition 5 *If $f : P \to Q$ is an isomorphism then $\mu_P(x) = \mu_Q(f(x))$.* $\blacksquare$

1. B_n is isomorphic as a poset to the n-fold product $(C_1)^n$ where a set corresponds to its bit-string. Now for each bit $\mu(0) = 1, \mu(1) = -1$, so $\mu(x) = (-1)^{|x|}$.

2. If n has prime factorization $n = \prod_i p_i^{n_i}$ then we have an isomorphism $D_n \cong \times_i C_{n_i}$. So $p_i^{n_i}$ contributes -1 or 0 to the product for μ depending on whether $n_i = 1$ or $n_i \geq 2$.

The Möbius Inversion Theorem (MIT)

It is convenient to extend the definition of μ. If $x \le y$ in P then we have the *interval*

$$[x, y] = \{z \in P \ : \ x \le z \le y\}.$$

Let $\mathrm{Int}\, P$ denote the set of intervals of P.

If P is any finite poset, then its *Möbius function*, $\mu : \mathrm{Int}\, P \to \mathbf{Z}$, is defined recursively by

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \le z < y} \mu(x, z) & \text{if } x < y. \end{cases}$$

Equivalently

$$\sum_{x \le z \le y} \mu(x, z) = \delta_{x,y}.$$

So if P has a $\hat{0}$ then $\mu(x) = \mu(\hat{0}, x)$. Furthermore, we can consider $[x, y]$ as a poset in it's own right with $x = \hat{0}_{[x,y]}$ and in this poset the new definition reduces to the old one.

Note that our previous computations also give the two-variable version of μ since in all cases an interval $[x, y]$ in P_n ($P = C$, B, or D) satisfies

$$[x, y] \cong P_k$$

for some $k \le n$.

Theorem 6 *Let P be a poset and $f, g : P \to \mathbb{C}$.*

MIT1 If for all $x \in P$ we have $f(x) = \sum_{y \leq x} g(y)$ then

$$g(x) = \sum_{y \leq x} \mu(y, x) f(y).$$

MIT2 If for all $x \in P$ we have $f(x) = \sum_{y \geq x} g(y)$ then

$$g(x) = \sum_{y \geq x} \mu(x, y) f(y).$$

Proof. We prove MIT2 as MIT1 is similar.

$$\begin{aligned}
\sum_{y \geq x} \mu(x, y) f(y) &= \sum_{y \geq x} \mu(x, y) \sum_{z \geq y} g(z) \\
&= \sum_{z \geq x} g(z) \sum_{x \leq y \leq z} \mu(x, y) \\
&= \sum_{z \in P} g(z) \delta_{x,z} \\
&= g(x). \qquad \blacksquare
\end{aligned}$$

Note (a) $\mathbb{C}$ above can be replaced by any vector space.

(b) The converses of MIT1 & 2 are also true.

It is now easy to obtain the three theorems from the introduction as corollaries by using Möbius inversion over D_n, B_n, and C_n, respectively.

I. If $f(m) = \sum_{d|m} g(d)$ for all m then in D_n we have $f(m) = \sum_{d \le m} g(d)$. So by MIT and $[d, m] \cong D_{m/d}$

$$g(m) = \sum_{d \le m} \mu(d, m) f(d) = \sum_{d|m} \mu(m/d) f(d).$$

II. For the PIE, given $S_1, \ldots, S_n \subseteq S$ and $x \in B_n$ let $S_x = \cap_{i \in x} S_i$. Define $f, g : B_n \to \mathbb{Z}_{\ge 0}$ by

$$\begin{aligned} f(x) &= |S_x|, \\ g(x) &= |S_x - \bigcup_{i \notin x} S_i|. \end{aligned}$$

For example in B_2

$$\begin{array}{ll} f(\emptyset) = |S| & g(\emptyset) = |S - (S_1 \cup S_2)| \\ f(1) = |S_1| & g(1) = |S_1 - S_2| \\ f(2) = |S_2| & g(2) = |S_2 - S_2| \\ f(12) = |S_1 \cap S_2| & g(12) = |S_1 \cap S_2|. \end{array}$$

By definition $f(x) = \sum_{y \ge x} g(y)$, so by MIT2

$$\left| S - \bigcup_{i=1}^{n} S_i \right| = g(\emptyset) = \sum_{y \ge \emptyset} \mu(y) f(y) = \sum_{y \in B_n} (-1)^{|y|} \left| \bigcap_{i \in y} S_i \right|.$$

III. For the FTDC, use $g(m) := Sf(m) = \sum_{i \le m} f(i)$ in C_n. So by MIT1 and $[i, j] \cong C_{j-i}$

$$f(m) = \sum_{i \le m} \mu(i, m) g(i) = g(m) - g(m-1)$$

$$= \Delta g(m) = \Delta S f(m).$$

The incidence algebra

Consider the *incidence algebra of P, $I(P)$*, which consists of all functions $f : \text{Int}(P) \to \mathbb{C}$. The multiplication in this algebra is *convolution* defined by

$$f * g(x,y) = \sum_{x \le z \le y} f(x,z)g(z,y).$$

It will be convenient to extend a $f \in I(P)$ to any pair $(x,y) \in P \times P$ by letting $f(x,y) = 0$ if $x \not\le y$. So

$$f * g(x,y) = \sum_{z \in P} f(x,z)g(z,y).$$

Let $x_1, x_2, \ldots, x_n$ be a fixed linear extension of P (so $P = \{x_1, \ldots, x_n\}$ and $x_i < x_j$ in P implies $i < j$). Define a set of complex matrices $\text{Mat}(P)$ with rows and columns indexed by $x_1, \ldots, x_n$:

$$M \in \text{Mat}(P) \iff m_{x,y} = 0 \text{ if } x \not\le y.$$

Proposition 7 *$I(P) \cong M(P)$ as algebras via the isomporphism*

$$f \in I(P) \leftrightarrow M_f = (f(x,y)). \quad \blacksquare$$

For example, B_2 has linear extension $\emptyset, 1, 2, 12$ and

$$M_\mu = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Corollary 8 *1. $I(P)$ has an identity element denoted $1 : \mathrm{Int}(P) \to \mathbb{C}$ given by $1(x, y) = \delta_{x,y}$.*

2. $f \in I(P)$ is invertible iff $f(x, x) \neq 0$ for all $x \in P$.

Proof. 1. M_1 is the identity matrix.

2. M_f is invertible iff $\det M_f \neq 0$. Since M_f is upper triangular, $\det M_f = \prod_{x \in P} f(x, x)$. ■

Define $\zeta \in I(P)$ by

$$\zeta(x, y) = 1 \quad \forall [x, y] \in \mathrm{Int}\, P.$$

Corollary 9 *We have $\zeta^{-1} = \mu$.*

Proof. It suffices to show $\mu * \zeta = 1$ but

$$(\mu * \zeta)(x, y) = \sum_{x \le z \le y} \mu(x, z)\zeta(z, y)$$

$$= \sum_{x \le z \le y} \mu(x, z) = 1(x, y). \quad ■$$

The Möbius algebra

A *lattice* is a poset L where every $x, y \in L$ have a greatest lower bound or meet, $x \wedge y$, and a least upper bound or joint, $x \vee y$. For example,

Poset	$x \wedge y$	$x \vee y$
C_n	$\min\{x, y\}$	$\max\{x, y\}$
B_n	$x \cap y$	$x \cup y$
D_n	$\gcd(x, y)$	$\operatorname{lcm}(x, y)$

Consider the complex vector space of formal sums

$$M(L) = \{\sum_{x \in L} c_x x \ : \ c_x \in \mathbb{C}\}.$$

So the elements of L are a basis for $M(L)$. For $x \in L$, define

$$\epsilon_x := \sum_{z \le x} \mu(z, x) x$$

in $M(L)$. For example in $A(B_2)$ where we have $\mu(z, x) = (-1)^{|x-z|}$

$$\begin{aligned}
\epsilon_\emptyset &= \mu(\emptyset, \emptyset)\emptyset = \emptyset, \\
\epsilon_1 &= \mu(1, 1)1 - \mu(\emptyset, 1)\emptyset = 1 - \emptyset, \\
\epsilon_2 &= \mu(2, 2)2 - \mu(\emptyset, 2)\emptyset = 2 - \emptyset, \\
\epsilon_{12} &= \sum_{x \in B_2} \mu(x, 12)x = 12 - 1 - 2 + \emptyset.
\end{aligned}$$

Lemma 10 *a. If* $x \in L$ *then* $x = \sum_{z \le x} \epsilon_z$.

b. The set $\mathcal{B} = \{\epsilon_x \ : \ x \in L\}$ *is a basis for* $M(L)$.

Proof. 1. This follows immediately by applying the converse of MIT1 to the definition of the ϵ_x.

2. Since $|\mathcal{B}| = |L|$ we need only prove that the ϵ_x span, which follows from part 1. ■

The *Möbius algebra* is $M(L)$ with multiplication

$$\epsilon_x \cdot \epsilon_y = \delta_{x,y}\epsilon_x.$$

Proposition 11 *If* $x, y \in L$ *then* $x \cdot y = x \wedge y$.

Proof. Using Lemma 10.a

$$\begin{aligned} x \cdot y &= \left(\sum_{z \le x} \epsilon_z\right) \cdot \left(\sum_{w \le y} \epsilon_w\right) \\ &= \sum_{\substack{z \le x \\ w \le y}} \epsilon_z \cdot \epsilon_w \\ &= \sum_{z \le x,y} \epsilon_z \\ &= \sum_{z \le x \wedge y} \epsilon_z \\ &= x \wedge y. \quad \blacksquare \end{aligned}$$

Theorem 12 (Wiesner's Theorem) *Suppose L is a finite lattice and $c \in L \setminus \hat{1}$. Then*

$$\sum_{x \wedge c = \hat{0}} \mu(x, \hat{1}) = 0.$$

Proof. Expand $c \cdot \epsilon_{\hat{1}}$ in two ways

$$c \cdot \epsilon_{\hat{1}} = \sum_{d \leq c} \epsilon_d \cdot \epsilon_{\hat{1}} = 0$$

since $d \leq c < \hat{1}$ so $d \neq \hat{1}$ and $\epsilon_d \cdot \epsilon_{\hat{1}} = 0$. Now by the Proposition

$$0 = c \cdot \epsilon_{\hat{1}} = c \cdot \sum_{x \leq \hat{1}} \mu(x, \hat{1}) x = \sum_{x \in L} \mu(x, \hat{1}) x \wedge c.$$

Taking the coefficient of $\hat{0}$ on both sides finishes the proof. ∎

Theorem 13 (Dual Weisner) *Suppose L is a finite lattice and $d \in L \setminus \hat{0}$. Then*

$$\sum_{x \vee c = \hat{1}} \mu(\hat{0}, x) = 0.$$

The *atoms* of poset P are

$$A = A(P) = \{a \in P \ : \ a \text{ covers } \hat{0}\}.$$

The *coatoms* of poset P are

$$A^* = A^*(P) = \{b \in P \ : \ \hat{1} \text{ covers } b\}.$$

For example

P	$A(P)$	$A^*(P)$
C_n	$\{1\}$	$\{n-1\}$
B_n	$\{x \ : \ \lvert x\rvert = 1\}$	$\{x \ : \ \lvert x\rvert = n-1\}$
D_n	$\{p \text{ prime} : \ p\lvert n\}$	$\{n/p : \ p \text{ prime}\}$

Theorem 14 (P. Hall) *Let L be a finite lattice.*

a. If $\bigvee A(L) \neq \hat{1}$ then $\mu(\hat{0}, \hat{1}) = 0$.

b. If $\bigwedge A^*(L) \neq \hat{0}$ then $\mu(\hat{0}, \hat{1}) = 0$.

Proof. a. Let $c = \bigvee A(L)$ in Weisner which is OK since $c \neq \hat{1}$.

$$\therefore \sum_{x \wedge c = \hat{0}} \mu(x, \hat{1}) = 0.$$

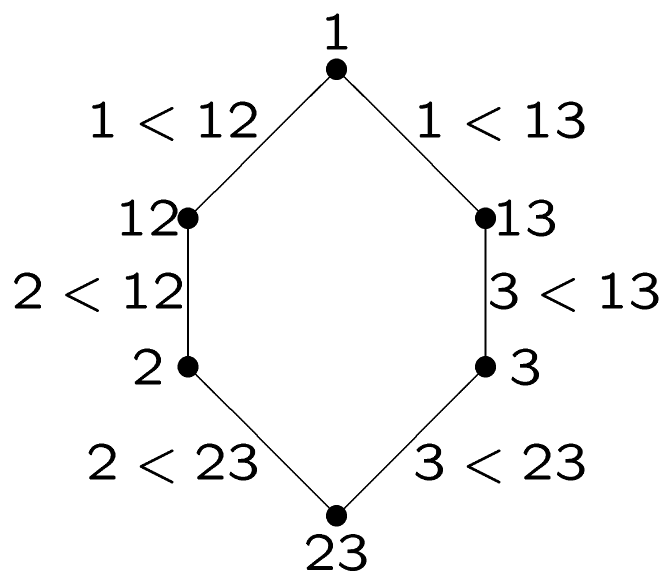
But $x \wedge c = \hat{0}$ implies $x = \hat{0}$. (If $x \neq \hat{0}$ then $x \geq a$ for some $a \in A(L)$ so $x \wedge c \geq a$.) But then the sum above has only one term corresponding to $x = \hat{0}$ and so reduces to $\mu(\hat{0}, \hat{1}) = 0$. ■

The order complex

A *chain* C *of length* l in P is $C : x_0 < x_1 < \ldots < x_l$. Equivalently $C \cong C_l$. Let P have a $\hat{0}$ and a $\hat{1} \neq \hat{0}$. The *order complex* is

$$\Delta P = \{C \;:\; x_0 > \hat{0},\; x_l < \hat{1}\}$$

Then ΔP is a simplicial complex since any subset of a chain is a chain. (We do not include $\hat{0}$ or $\hat{1}$ in our chains since if we did ΔP would be contractible.) For example



The simplices of ΔP of dimension l are

$$\Delta^l P = \{C \in \Delta P \;:\; C \text{ has length } l\}.$$

Also $\Delta^{-1} P = \{\emptyset\}$ and $\Delta^l P = \emptyset$ for $l \leq -2$.

Lemma 15 *For* $l \geq 0$: $(\zeta-1)^l(\hat{0},\hat{1}) = |\Delta^{l-2}P|$.

Proof. From the definitions

$$(\zeta-1)(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x = y \end{cases}$$

$$\begin{aligned} \therefore (\zeta-1)^l(\hat{0},\hat{1}) \\ &= \sum_{\hat{0} \leq x_0 \leq \ldots \leq x_{l-2} \leq \hat{1}} (\zeta-1)(\hat{0},x_0)\cdots(\zeta-1)(x_{l-2},\hat{1}) \\ &= \sum_{\hat{0} < x_0 < \ldots < x_{l-2} < \hat{1}} 1 \\ &= |\Delta^{l-2}P|. \quad \blacksquare \end{aligned}$$

Proposition 16 (Reduced Euler Characteristic)

$$\mu(\hat{0},\hat{1}) = \sum_{l \geq -1} (-1)^l |\Delta^l P|.$$

Proof. Using the lemma and $(\zeta-1)(\hat{0},\hat{1}) = 0$:

$$\begin{aligned} \mu(\hat{0},\hat{1}) &= \zeta^{-1}(\hat{0},\hat{1}) \\ &= [1+(\zeta-1)]^{-1}(\hat{0},\hat{1}) \\ &= \sum_{l \geq 0} (-1)^l (\zeta-1)^l(\hat{0},\hat{1}) \\ &= \sum_{l \geq 0} (-1)^l |\Delta^{l-2}P| \\ &= \sum_{l \geq -1} (-1)^l |\Delta^l P|. \quad \blacksquare \end{aligned}$$