Monotonic Sequence Games

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan
and
The Otago Theory Group
Department of Computer Science
University of Otago
Dunedin, New Zealand
www.cs.otago.ac.nz

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The Game on an Interval

The Game on the Rationals

Eine Kleine Game Theory

Open Questions

Outline

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and

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Ex. If $m = 2$ and $n = 3$ then $mn + 1 = 7$. A permutation in S_7, please!!

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Theorem

The winner of the game on $[mn+1]$ where $m \leq n$ is:

$m \backslash n$	0	1	2	3	4	5	6	7
0	A	A	A	A	A	A	A	A
1		B	A	B	A	B	A	B
2			A	A	A	A	A	A
3				A	A	A	A	?
4					A	?	?	?

where the patterns continue in each of the first 3 rows. ■

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1. Initially $I = \epsilon$, the empty sequence.
2. If $I = y_1y_2\ldots$ when x_i is picked, have x_i replace the smallest $y_j > x_i$ or append x_i to the right end of I if no such y_j exists.

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Ex. $\pi = 4\ 2$

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Ex. $\pi = 4\ 2\ 5$

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Ex. $\pi = 4\ 2\ 5\ 3\ 1$

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Theorem (Schensted, 1961)

If x_i is placed in column j of I,

$$j = \text{length of a longest increasing subsequence ending at } x_i,$$

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Since 6 was placed in the third column of I we have an increasing subsequence of length three ending at 6, e.g., 2 3 6.

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Theorem (Schensted, 1961)

If x_i is placed in column j of I, and in column k of D

$j =$ *length of a longest increasing subsequence ending at* x_i,
$k =$ *length of a longest decreasing subsequence ending at* x_i.

Similarly build a *decreasing list D* by reversing the inequalities.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

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Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

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D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

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Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

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Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

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D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

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$$
\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\
D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1
\end{array}
$$

$$
C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}, \quad \overset{2}{P}\ \overset{5}{P},
$$

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lclcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B}$,	$\overset{2}{P}\ \overset{5}{P}$,	$\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,		

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lclcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B}$,	$\overset{2}{P}\ \overset{5}{P}$,	$\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}$,	

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lclcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$
\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\
D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\
C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}
\end{array}
$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\ C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P} \end{array}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} \quad x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}, \quad \overset{2}{P}\ \overset{5}{P}, \quad \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, \quad \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, \quad \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lllll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B}$,	$\overset{2}{P}\ \overset{5}{P}$,	$\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lllll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$
\begin{array}{llllllll}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \\
C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\,\overset{4}{B}, & \overset{2}{P}\,\overset{5}{P}, & \overset{2}{R}\,\overset{3}{P}\,\overset{5}{B}, & \overset{1}{P}\,\overset{3}{P}\,\overset{5}{B}, & \overset{1}{P}\,\overset{3}{P}\,\overset{6}{P}
\end{array}
$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{llll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$
\begin{array}{llllllll}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1
\end{array}
$$

$$
C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\,\overset{4}{B}, \quad \overset{2}{P}\;\overset{5}{P}, \quad \overset{2}{R}\,\overset{3}{P}\,\overset{5}{B}, \quad \overset{1}{P}\,\overset{3}{P}\,\overset{5}{B}, \quad \overset{1}{P}\,\overset{3}{P}\,\overset{6}{P}
$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lll} x_i \in I \quad \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I \quad \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I \quad \text{and} \quad x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

C : ϵ,

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in I & \Longrightarrow \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \uparrow \overset{4}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} \quad x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad {}_{\uparrow}\overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\ ,$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C:\quad \epsilon,\quad \uparrow\overset{4}{P},\quad \overset{2}{P}\ \overset{4}{B}\uparrow,$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{aligned} x_i \in I \quad \text{and} \quad x_i \notin D &\implies \text{color } x_i \text{ with } R, \\ x_i \notin I \quad \text{and} \quad x_i \in D &\implies \text{color } x_i \text{ with } B, \\ x_i \in I \quad \text{and} \quad x_i \in I &\implies \text{color } x_i \text{ with } P. \end{aligned}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\uparrow \overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B}\ \uparrow$,	$\overset{2}{P}\ \overset{5}{P}$,			

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lclcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

C : ϵ, $\quad {\uparrow}\overset{4}{P}$, $\quad \overset{2}{P}\ \overset{4}{B}{\uparrow}$, $\quad \overset{2}{P}{\uparrow}\overset{5}{P}$,

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lcl}
x_i \in I \quad \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I \quad \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I \quad \text{and} \quad x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\uparrow \overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B} \uparrow$,	$\overset{2}{P} \uparrow \overset{5}{P}$,	$\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,		

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lllll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$
\begin{array}{llllllll}
I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\
D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1
\end{array}
$$

$$
C: \quad \epsilon, \quad \uparrow \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B} \uparrow, \quad \overset{2}{P} \uparrow \overset{5}{P}, \quad \uparrow \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B},
$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lcllcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$
\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \\
C: & \epsilon, & \uparrow\overset{4}{P}, & \overset{2}{P}\,\overset{4}{B}\uparrow, & \overset{2}{P}\uparrow\overset{5}{P}, & \uparrow\overset{2}{R}\,\overset{3}{P}\,\overset{5}{B}, & \overset{1}{P}\,\overset{3}{P}\,\overset{5}{B}\,, &
\end{array}
$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{lclcl}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in I & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\uparrow\overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B}\uparrow$,	$\overset{2}{P}\uparrow\overset{5}{P}$,	$\uparrow\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}\uparrow$,	

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
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D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\uparrow P^{4}$,	$P^{2}\,B^{4}\uparrow$,	$P^{2}\uparrow P^{5}$,	$\uparrow R^{2}\,P^{3}\,B^{5}$,	$P^{1}\,P^{3}\,B^{5}\uparrow$,	$P^{1}\,P^{3}\,P^{6}$

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Theorem (Otago-S)

The winner of the game on $\mathbb{Q}$ where $m \leq n$ is:

$m \backslash n$	0	1	2	3	4	5	6	7
0	A	A	A	A	A	A	A	A
1		B	B	B	B	B	B	B
2			A	B	A	B	A	B
3				A	A	A	A	A
4					A	A	A	A

where the patterns continue in each of the first 5 rows. ■

Outline

The Game on an Interval

The Game on the Rationals

Eine Kleine Game Theory

Open Questions

Let G be a 2-person, “last player to move wins” game with no draw positions.

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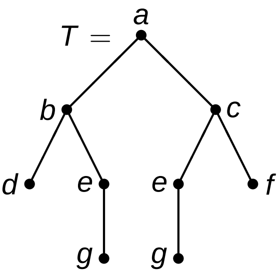
$T =$ 

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If the same position w is a child of both v and v' then identify the copies of w to get a (di)graph $\overline{T}$.

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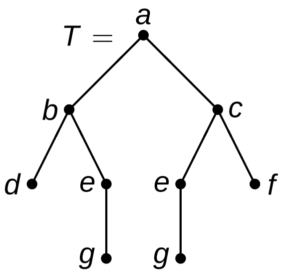

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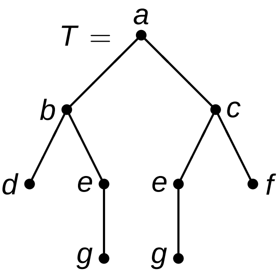

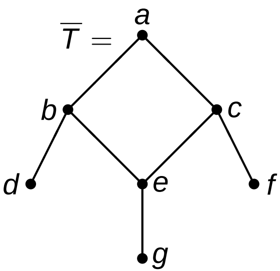

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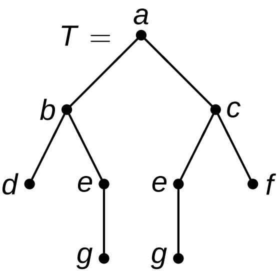

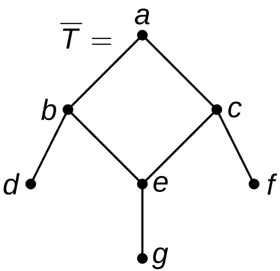

Let $\ell(v) = \begin{cases} \mathcal{N} & \text{if the next player wins from position } v, \\ \mathcal{P} & \text{if the previous player wins from position } v. \end{cases}$

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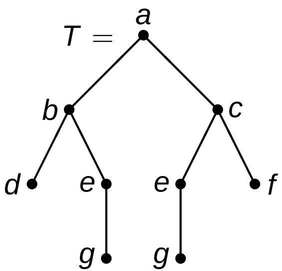

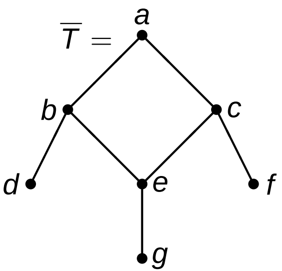

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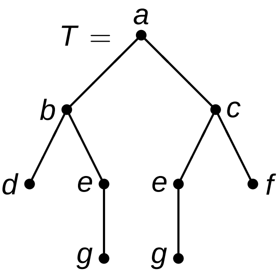

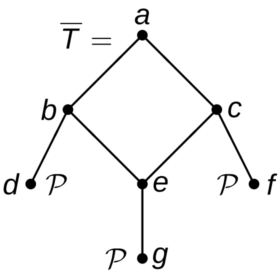

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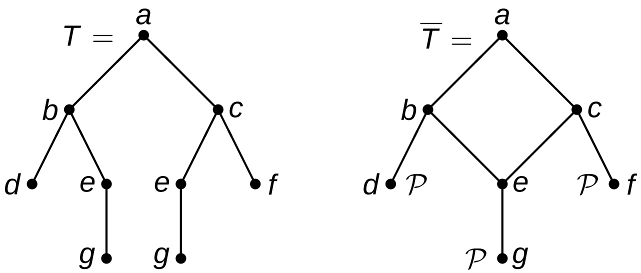

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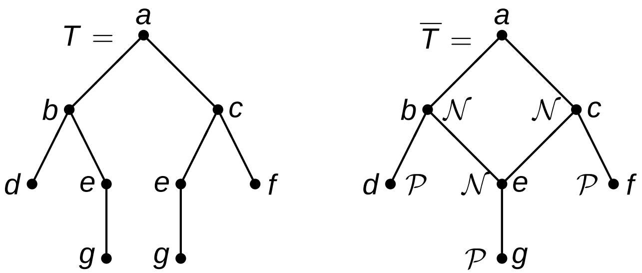

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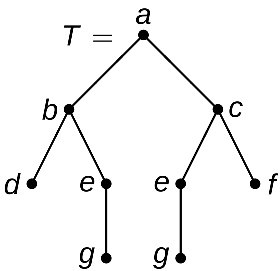

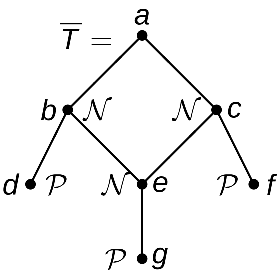

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(i) if there is a $\mathcal{P}$-child of v then let $\ell(v) = \mathcal{N}$

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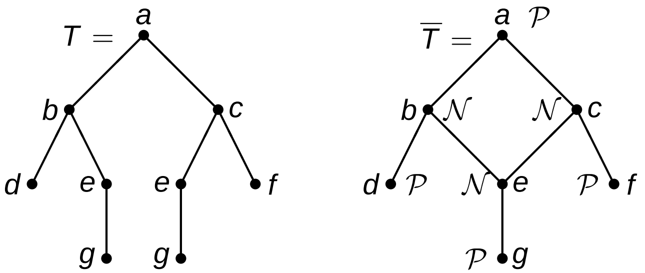

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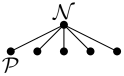
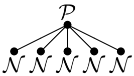
(i) 
$\mathcal{N}$
$\mathcal{P}$

(ii) 
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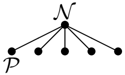
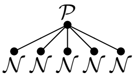
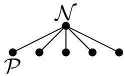
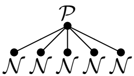
(i) 

(ii) 

Theorem (Otago-S)

The winner of the game on $\mathbb{Q}$ where $m = n \geq 3$ is A.

(i) 

(ii) 

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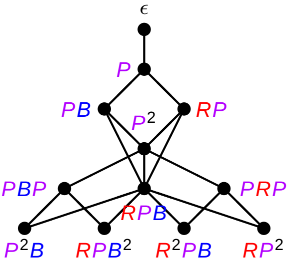
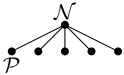
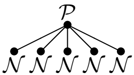


(i) 

(ii) 

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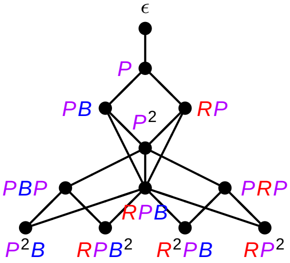
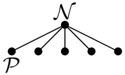
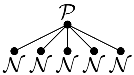


Suppose B wins, so $\ell(\epsilon) = \mathcal{P}$.

(i) 

(ii) 

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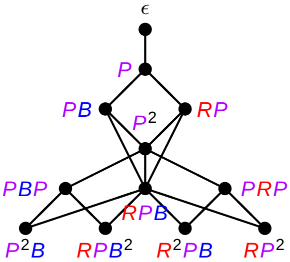

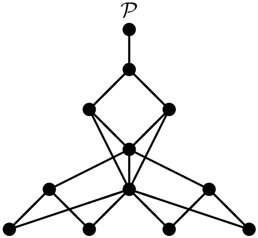

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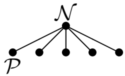
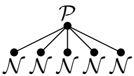
(i) 

(ii) 

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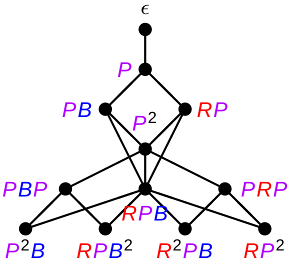
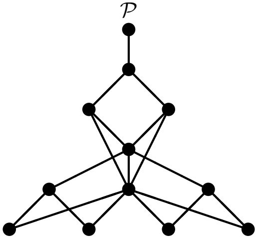
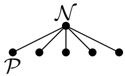




Suppose B wins, so $\ell(\epsilon) = \mathcal{P}$. So $\ell(P) = \mathcal{N}$ by (ii).

(i)



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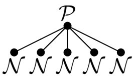

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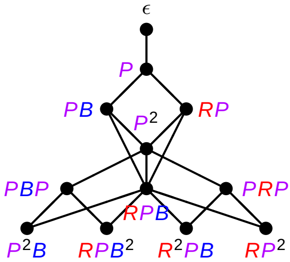

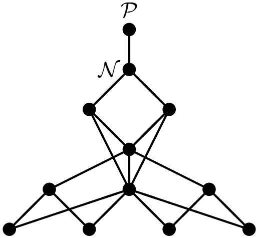

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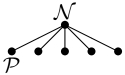
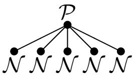
(i) 

(ii) 

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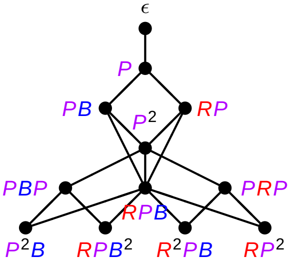
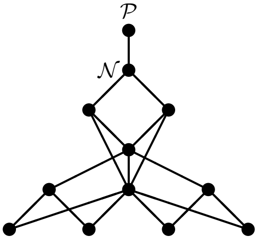
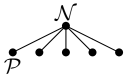
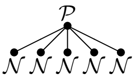




Suppose B wins, so $\ell(\epsilon) = \mathcal{P}$. So $\ell(P) = \mathcal{N}$ by (ii). So $\ell(PB) = \mathcal{P}$ or $\ell(RP) = \mathcal{P}$ by (i).

(i) 

(ii) 

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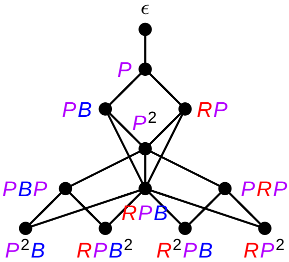

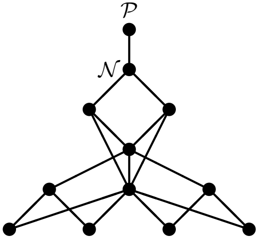

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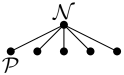
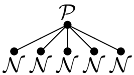
(i) 

(ii) 

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The winner of the game on $\mathbb{Q}$ where $m = n \geq 3$ is A.

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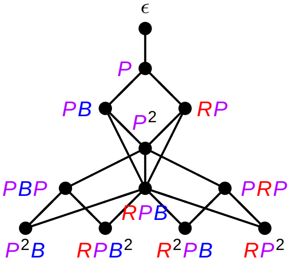

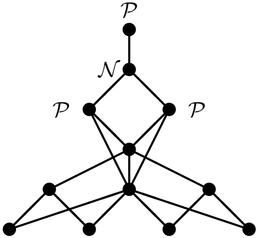

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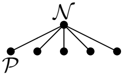
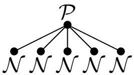
(i) 

(ii) 

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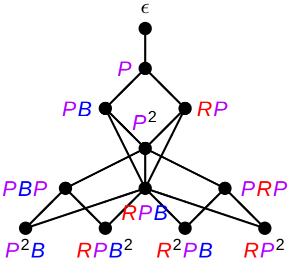

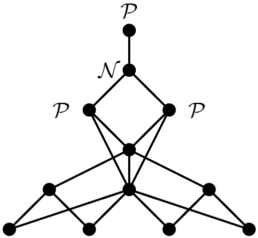

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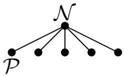
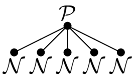
(i) 

(ii) 

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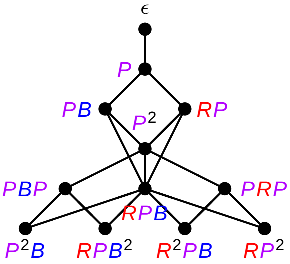

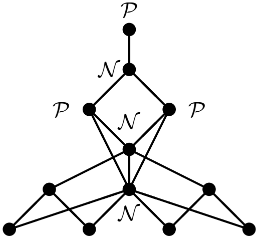

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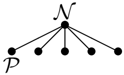
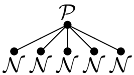
(i) 

(ii) 

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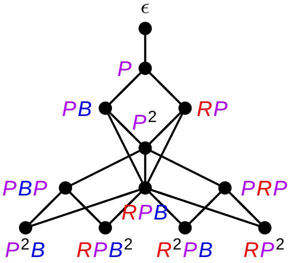

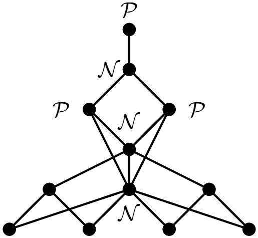

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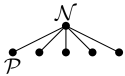
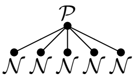
(i) 

(ii) 

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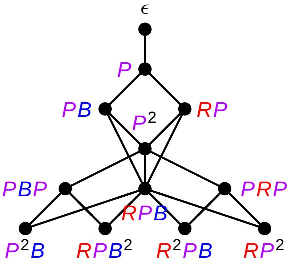

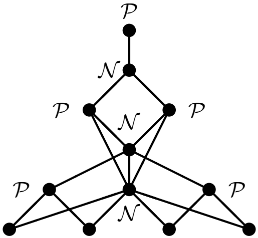

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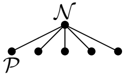
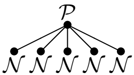
(i) 

(ii) 

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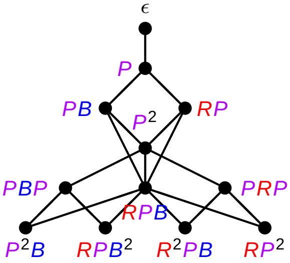

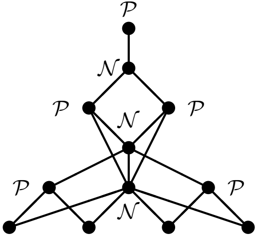

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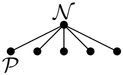
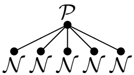
(i) 

(ii) 

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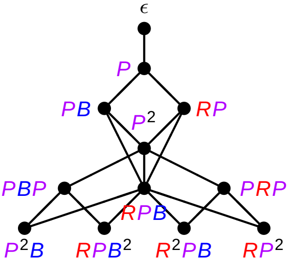

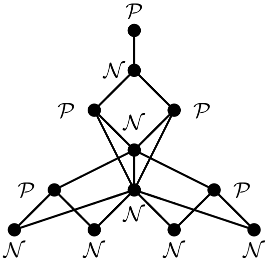

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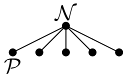
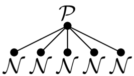
(i) 

(ii) 

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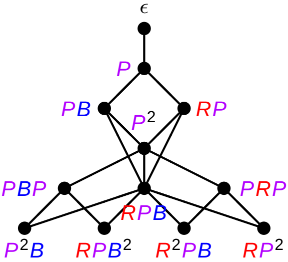

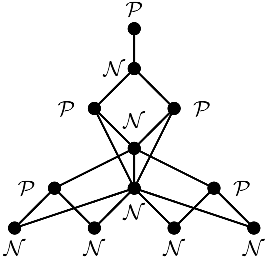

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Outline

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The Game on the Rationals

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