Mahonian Pairs

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Outline

Mahonian pairs and Foata's bijection

Integer partitions and a Catalan pair

An application

The past, the future, and open problems

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Theorem (MacMahon)

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If $b_i > a_{i+1}$, then obtain w_{i+1} similarly interchanging $\le$ and $>$.

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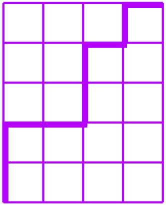

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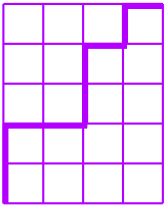

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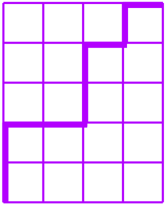

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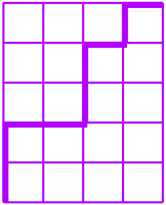

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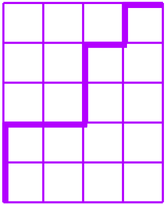

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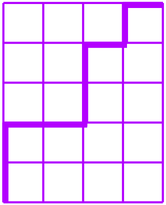

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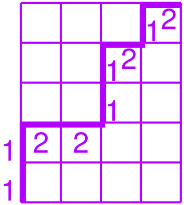

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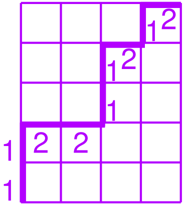

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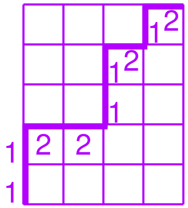

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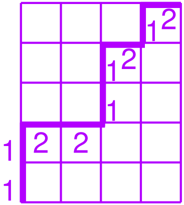

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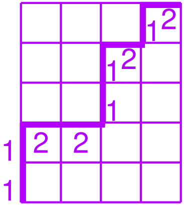

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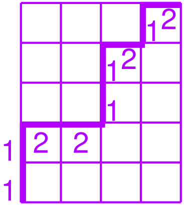

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If $v \in \{1, 2\}^*$ and $\lambda = \lambda(\phi(v))$, then

1. $\operatorname{maj} v = |\lambda|$, *and*
2. $\operatorname{des} v = d(\lambda)$ *where* $\operatorname{des} v = \#\{i : a_i > a_{i+1}\}$.

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and so

$$\sum_{v \in B_n} q^{\operatorname{maj} v} t^{\operatorname{des} v} = \sum_{\lambda \in R_n} q^{|\lambda|} t^{d(\lambda)}.$$

Outline

Mahonian pairs and Foata's bijection

Integer partitions and a Catalan pair

An application

The past, the future, and open problems

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We define the map $CSV' : P'_{\neq 1} \to R'_{>0}$ by $CSV'(\lambda') = \mu'$ iff $CSV(\lambda) = \mu$.

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Theorem (S-Savage)

We have

$$CSV' = \phi \circ GK \circ \phi^{-1}.$$

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THANKS FOR LISTENING!