Maximal Independent Sets In Graphs With At Most r Cycles

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1. Introduction

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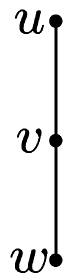
1. Introduction

Let $G = (V, E)$ be a simple graph. A subset $I \subseteq V$ is *independent* if there are no edges between any two elements of I. Also, I is *maximal* if it is contained in no larger independent set. We let

$m(G) =$ number of maximal independent sets in G.

Ex. The path P_3

$P_3 =$ 

has independent sets

$$I_1 = \{u, w\}, \quad I_2 = \{v\}$$

so $m(P_3) = 2$.

Given a family $\mathcal{F}$ of graphs with $|V| = n$, we ask two questions

1. What is $\max_{G \in \mathcal{F}} m(G)$?

2. What are the extremal graphs?

1a. History

Given a family $\mathcal{F}$ of graphs with $|V| = n$, we ask two questions

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2. What are the extremal graphs?

Erdös and Moser possed the original question for all graphs with n vertices in the early 1960s. (Actually, they posed it in terms of maximal cliques.) Below is a history of the questions that have been addressed concerning graphs with a bounded number of cycles.

$\mathcal{F}$	Who	When	Extremal?
all graphs	Erdös	~1960	Y
all graphs	Moon and Moser	1965	Y
connected	Füredi	1987	N
connected	Griggs, Grinstead, and Guichard	1988	Y
trees	Wilf	1986	N
trees	Sagan	1988	Y
≤ 1 cycle	Jou and Chang	1997	Y
$\leq 2, 3$ cycles	Goh and Koh	2001	Y
$\leq r$ cycles	Sagan and Vatter	2001	Y

2. Previous Results

Let tG stand for the disjoint union of t copies of G. Define

$$G(n) := \begin{cases} \frac{n}{3}K_3 & \text{if } n \equiv 0 \pmod 3, \\ 2K_2 \uplus \frac{n-4}{3}K_3 & \text{if } n \equiv 1 \pmod 3, \\ K_2 \uplus \frac{n-2}{3}K_3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Also let

$$G'(n) := K_4 \uplus \tfrac{n-4}{3}K_3 \text{ if } n \equiv 1 \pmod 3.$$

Since $m(G \uplus H) = m(G)m(H)$, we see that

$$g(n) := m(G(n)) = \begin{cases} 3^{\frac{n}{3}} & \text{if } n \equiv 0 \pmod 3, \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod 3, \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Note that $m(G'(n)) = m(G(n))$.

Theorem 1 (Moon and Moser) *Let G be a graph with $n \geq 2$ vertices. Then*

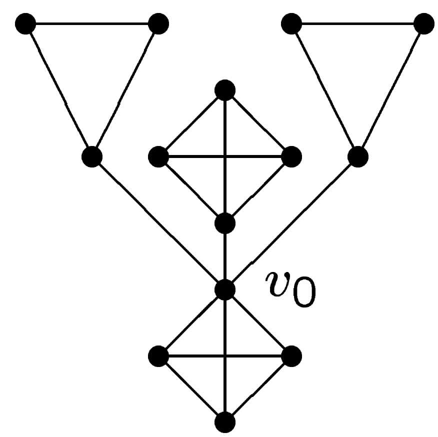
$$m(G) \leq g(n)$$

with equality if and only if $G = G(n)$ or $G'(n)$. ■

Construct a graph $K_m * G$ by picking a vertex v_0 in K_m and connecting it to a single vertex in each component of G. If $n \geq 6$ then let

$$C(n) := \begin{cases} K_3 * \frac{n-3}{3} K_3 & \text{if } n \equiv 0 \pmod 3, \\ K_4 * \frac{n-4}{3} K_3 & \text{if } n \equiv 1 \pmod 3, \\ K_4 * \left(K_4 \uplus \frac{n-8}{3} K_3 \right) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

The graph $C(14)$ is



Letting $c(n) := m(C(n))$ we have

$$c(n) = \begin{cases} 2 \cdot 3^{\frac{n-3}{3}} + 2^{\frac{n-3}{3}} & \text{if } n \equiv 0 \pmod 3, \\ 3^{\frac{n-1}{3}} + 2^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod 3, \\ 4 \cdot 3^{\frac{n-5}{3}} + 3 \cdot 2^{\frac{n-8}{3}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem 2 (Griggs, Grinstead, and Guichard) *Let G be a connected graph with $n \geq 6$ vertices. Then*

$$m(G) \leq c(n)$$

with equality if and only if $G = C(n)$. ■

The Main Theorem

Now suppose that $n \geq 3r - 1$. Define

$$G(n,r) := \begin{cases} rK_3 \uplus \frac{n-3r}{2} K_2 & \text{if } n \equiv r \pmod 2, \\ (r-1)K_3 \uplus \frac{n-3r+3}{2} K_2 & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Further, let $g(n,r) := m(G(n,r))$ and so

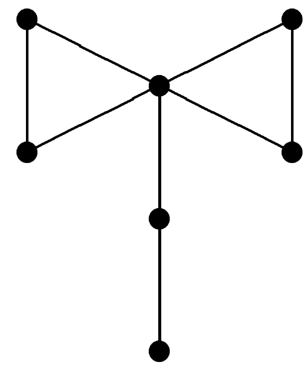
$$g(n,r) = \begin{cases} 3^r \cdot 2^{\frac{n-3r}{2}} & \text{if } n \equiv r \pmod 2, \\ 3^{r-1} \cdot 2^{\frac{n-3r+3}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Now assume $n \geq 3r$ and define

$$C(n,r) := \begin{cases} K_3 * \left((r-1)K_3 \uplus \frac{n-3r}{2} K_2\right) & \text{if } n \equiv r \pmod 2, \\ K_1 * \left(rK_3 \uplus \frac{n-3r-1}{2} K_2\right) & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

As usual, we let $c(n,r) := m(C(n,r))$ and compute

$$c(n,r) = \begin{cases} 3^{r-1} \cdot 2^{\frac{n-3r+2}{2}} + 2^{r-1} & \text{if } n \equiv r \pmod 2, \\ 3^r \cdot 2^{\frac{n-3r-1}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Define E to be the graph 

Theorem 3 (Sagan and Vatter) *Let G be a graph with n vertices and at most r cycles where $r \geq 1$.*

(I) If $n \geq 3r - 1$ then for all such graphs we have

$$m(G) \leq g(n, r)$$

with equality if and only if $G \cong G(n, r)$.

(II) If $n \geq 3r$ then for all such graphs which are connected we have

$$m(G) \leq c(n, r)$$

with equality if and only if $G \cong C(n, r)$ or if G is one of the following exceptional cases

n	r	possible $G \not\cong C(n, r)$
4	1	P_4
5	1	C_5
7	2	$C(7, 1), E$

Proof (sketch) Consider the case of all graphs with n vertices and at most r cycles, $n \geq 3r-1$. Double induct on r and n, where the cases $r \leq 3$ have been done previously. If $n = 3r-1$ or $3r$ then

$$g(n,r) = g(n) \quad \text{and} \quad G(n,r) = G(n)$$

so we are done by Moon-Moser.

One can show that if G has two or more intersecting cycles then $m(G) < g(n,r)$, so it suffices to look at graphs all of whose blocks are K_2 or cycles. Let B be an endblock of G. We have three cases depending on whether $B \cong K_2$, K_3, or C_p for $p \geq 4$.

Let's do $B \cong K_2$. For any complete endblock B

$$m(G) = \sum_{v \in V(B)} m(G - N[v])$$

where $N[v]$ is the closed neighborhood of v. Also, $g(n,r)$ is monotonically inceasing in n. Now supposing $V(B) = \{v, w\}$, then both $G - N[v]$ and $G - N[w]$ have at most $n-2$ vertices and at most r cycles. By induction and the above facts

$$m(G) \leq 2g(n-2,r) = g(n,r)$$

with equality iff $G - N[v] = G - N[w] \cong G(n-2,r)$. It follows that B is actually a component of G isomorphic to K_2 and so $G \cong G(n,r)$. ∎