Möbius Functions of Posets V: GCD Matrices

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Outline

Details about this work can be found in

GCD matrices, posets, and nonintersecting paths (with E. Altinisik, N. Tuglu), *Linear and Multilinear Alg.* **53** (2005) 75–84.

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So $M = (M^\alpha)^t M^\beta$, implying $\det M = \det M^\alpha \det M^\beta$. By triangularity of M^α, M^β we have $\det M = \prod_z \alpha(z,z)\beta(z,z)$. □

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Theorem (Wilf, 1968)

Let $f : P \to \mathbb{R}$ where P is a meet semi-lattice and let M be the matrix with $M_{x,y} = f(x \wedge y)$. Then

$$\det M = \prod_{z \in P} g(z)$$

where $g(z) = \sum_{w \le z} \mu(w,z) f(w)$.

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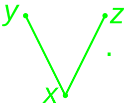

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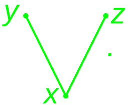

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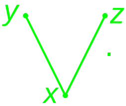
Example. Let $P =$ 

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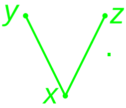
Example. Let $P =$ 

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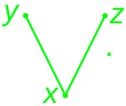
Example. Let $P =$ 

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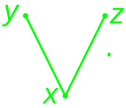
Example. Let $P =$ 

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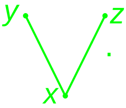
Example. Let $P =$ 

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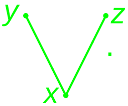
Example. Let $P =$ 

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$$g(x)g(y)g(z) = f(x)\,[f(y) - f(x)]\,[f(z) - f(x)]$$

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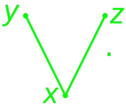
Example. Let $P =$ 

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$$\det M = f(x)f(y)f(z) + 2f(x)^3 - f(x)^3 - f(x)^2 f(y) - f(x)^2 f(z).$$

On the other hand

$$\begin{aligned} g(x) &= \mu(x,x)f(x) = f(x), \\ g(y) &= \mu(y,y)f(y) + \mu(x,y)f(x) = f(y) - f(x), \\ g(z) &= \mu(z,z)f(z) + \mu(x,z)f(x) = f(z) - f(x). \end{aligned}$$

$$g(x)g(y)g(z) = f(x)\,[f(y) - f(x)]\,[f(z) - f(x)] = \det M.$$

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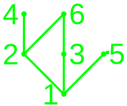
Example. $E_6 =$ 

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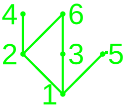
Example. $E_6 =$ 

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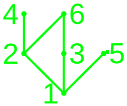
Example. $E_6 =$ 

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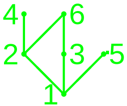
Example. $E_6 =$ 

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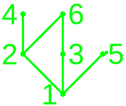
Example. $E_6 =$ 

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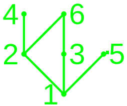
Example. $E_6 =$ 

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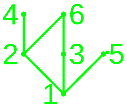
Example. $E_6 =$ 

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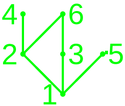
Example. $E_6 =$ 

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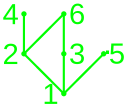

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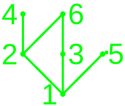
Example. $E_6 =$ 

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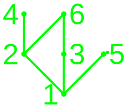
Example. $E_6 =$ 

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ATÉ À PRÓXIMA!!