Möbius Functions of Posets IV: Why the Characteristic Polynomial Factors

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

June 28, 2007

The Characteristic Polynomial

The Chromatic Polynomial

The Bond Lattice

The Connection

Outline

The Characteristic Polynomial

The Chromatic Polynomial

The Bond Lattice

The Connection

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$. Note that

$$\operatorname{rk} P = \operatorname{rk} \hat{1}.$$

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\mathrm{rk}\, x$. Note that

$$\mathrm{rk}\, P = \mathrm{rk}\, \hat{1}.$$

Example. In our four running example posets:

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$. Note that

$$\operatorname{rk} P = \operatorname{rk} \hat{1}.$$

Example. In our four running example posets:

1. $i \in C_n$: $\quad \operatorname{rk} i = i$.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$. Note that

$$\operatorname{rk} P = \operatorname{rk} \hat{1}.$$

Example. In our four running example posets:

1. $i \in C_n$: $\operatorname{rk} i = i$.
2. $S \in B_n$: $\operatorname{rk} S = |S|$.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$. Note that

$$\operatorname{rk} P = \operatorname{rk} \hat{1}.$$

Example. In our four running example posets:

1. $i \in C_n$: $\operatorname{rk} i = i$.
2. $S \in B_n$: $\operatorname{rk} S = |S|$.
3. $d \in D_n$: $\operatorname{rk} d = \sum_i m_i$ where $d = \prod_i p_i^{m_i}$.

For more information on this topic, see

B. Sagan, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.* **36** (1999), 113–134.

Let P be a graded poset and let $x \in P$. Then all saturated $\hat{0}$–x chains have the same length called the *rank of x* and denoted $\operatorname{rk} x$. Note that

$$\operatorname{rk} P = \operatorname{rk} \hat{1}.$$

Example. In our four running example posets:

1. $i \in C_n$: $\operatorname{rk} i = i$.
2. $S \in B_n$: $\operatorname{rk} S = |S|$.
3. $d \in D_n$: $\operatorname{rk} d = \sum_i m_i$ where $d = \prod_i p_i^{m_i}$.
4. $\pi = B_1/\ldots/B_k \in \Pi_n$: $\operatorname{rk} \pi = n - k$.

Let P be a graded poset and let t be a variable.

Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk}\hat{1} - \operatorname{rk} x}.$$

Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P; t) = \sum_{x \in P} \mu(x) t^{\mathrm{rk}\,\hat{1} - \mathrm{rk}\,x}.$$

Note that we use the corank, $\mathrm{rk}\,\hat{1} - \mathrm{rk}\,x$, as the exponent on t rather than the rank so as to make $q(P; t)$ monic:

Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk} \hat{1} - \operatorname{rk} x}.$$

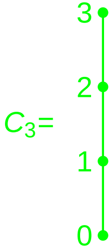
Note that we use the corank, $\operatorname{rk} \hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk} \hat{1} - \operatorname{rk} x}.$$

Note that we use the corank, $\operatorname{rk} \hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.

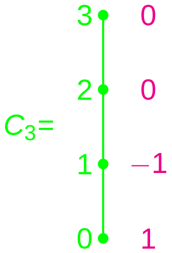


Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk} \hat{1} - \operatorname{rk} x}.$$

Note that we use the corank, $\operatorname{rk} \hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.

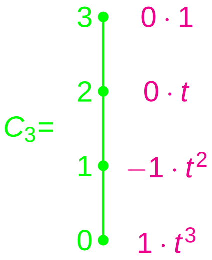


Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk} \hat{1} - \operatorname{rk} x}.$$

Note that we use the corank, $\operatorname{rk} \hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.



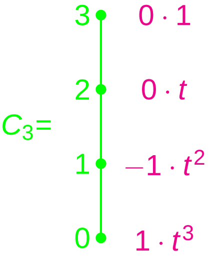
Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk}\hat{1} - \operatorname{rk} x}.$$

Note that we use the corank, $\operatorname{rk}\hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.

$$q(C_3) = t^3 - t^2 + 0 + 0$$



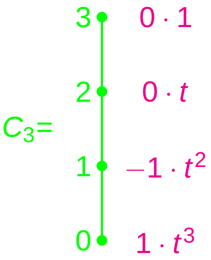
Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\operatorname{rk} \hat{1} - \operatorname{rk} x}.$$

Note that we use the corank, $\operatorname{rk} \hat{1} - \operatorname{rk} x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.

$$q(C_3) = t^3 - t^2 + 0 + 0 = t^2(t-1)$$



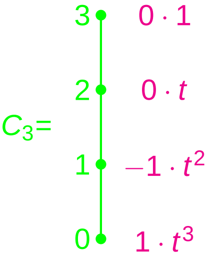
Let P be a graded poset and let t be a variable. The *characteristic polynomial of P* is the generating function

$$q(P) = q(P;t) = \sum_{x \in P} \mu(x) t^{\mathrm{rk}\,\hat{1} - \mathrm{rk}\,x}.$$

Note that we use the corank, $\mathrm{rk}\,\hat{1} - \mathrm{rk}\,x$, as the exponent on t rather than the rank so as to make $q(P;t)$ monic: the largest power of t is when $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Example.

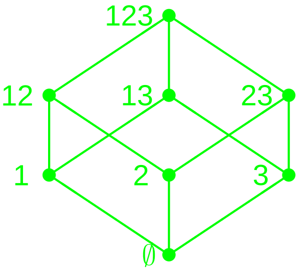
$$q(C_3) = t^3 - t^2 + 0 + 0 = t^2(t-1)$$



In general, $q(C_n;t) = t^{n-1}(t-1)$ (easy to verify).

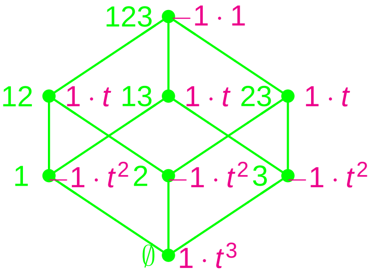
Example.

$B_3 =$



Example.

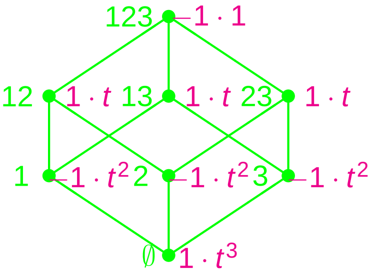
$B_3 =$



Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1$$

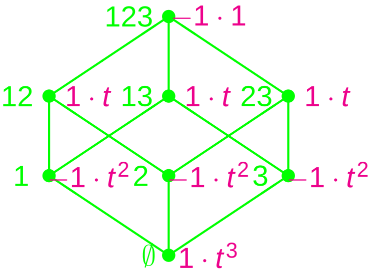
$B_3 =$



Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1$$
$$= (t-1)^3$$

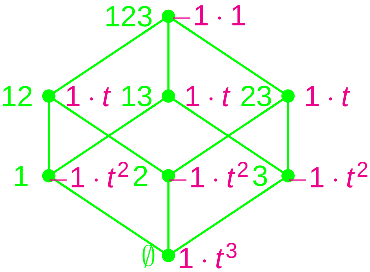
$B_3 =$



Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1$$
$$= (t-1)^3$$

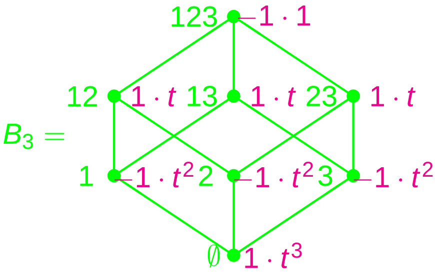
$B_3 =$



In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

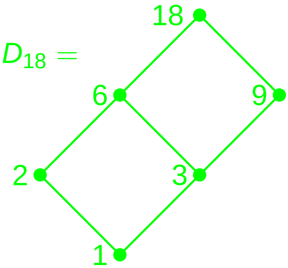
Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1$$
$$= (t-1)^3$$



In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

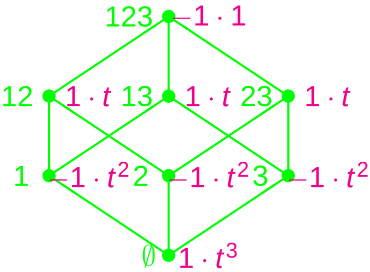
Example.



Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1$$
$$= (t-1)^3$$

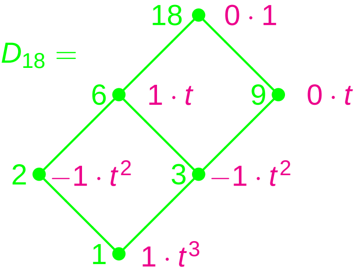
$B_3 =$



In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

Example.

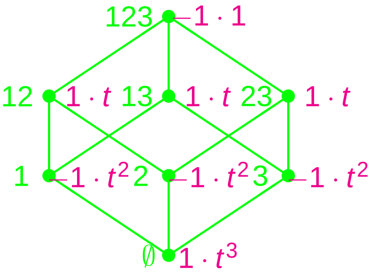
$D_{18} =$



Example.

$$q(B_3) = t^3 - 3t^2 + 3t - 1 = (t-1)^3$$

$B_3 =$

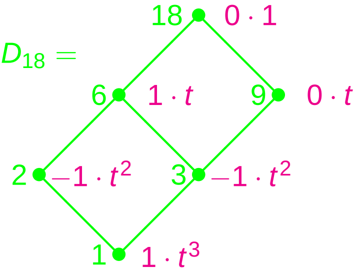


In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

Example.

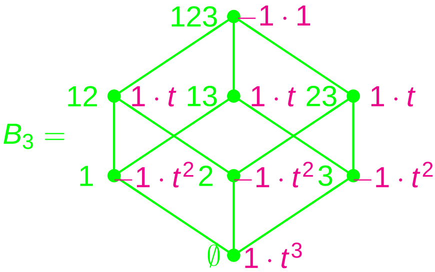
$$q(D_{18}) = t^3 - 2t^2 + t$$

$D_{18} =$



Example.

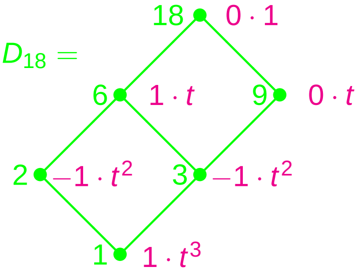
$$q(B_3) = t^3 - 3t^2 + 3t - 1$$
$$= (t-1)^3$$



In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

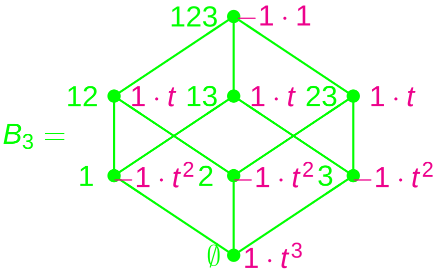
Example.

$$q(D_{18}) = t^3 - 2t^2 + t$$
$$= t(t-1)^2$$



Example.

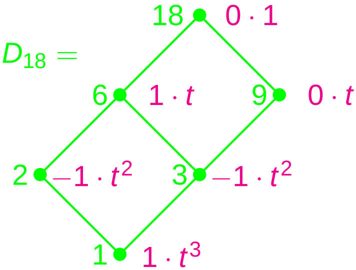
$$q(B_3) = t^3 - 3t^2 + 3t - 1 = (t-1)^3$$



In general $q(B_n; t) = (t-1)^n$ (use the Binomial Theorem).

Example.

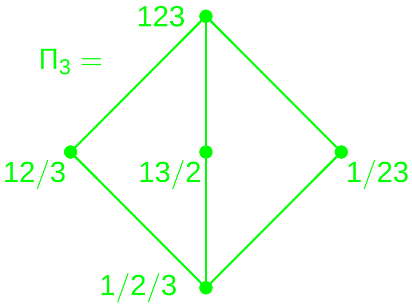
$$q(D_{18}) = t^3 - 2t^2 + t = t(t-1)^2$$



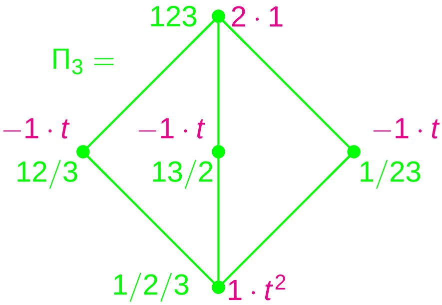
In general $q(D_n; t) = \prod_i t^{m_i - 1}(t-1)$ where $n = \prod_i p_i^{m_i}$.

Example.

$\Pi_3 =$

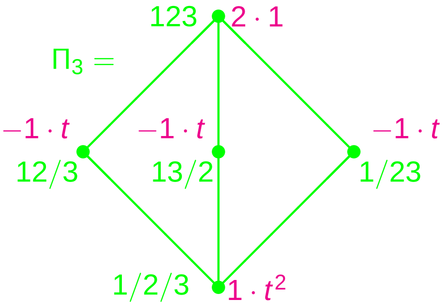


Example.



Example.

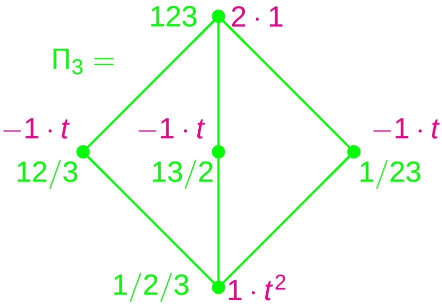
$q(\Pi_3) = t^2 - 3t + 2$



Example.

$$q(\Pi_3) = t^2 - 3t + 2$$
$$= (t-1)(t-2)$$

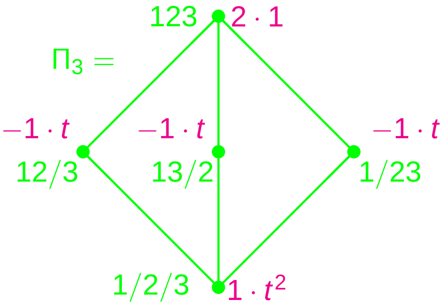
$\Pi_3 =$



Example.

$$q(\Pi_3) = t^2 - 3t + 2$$
$$= (t-1)(t-2)$$

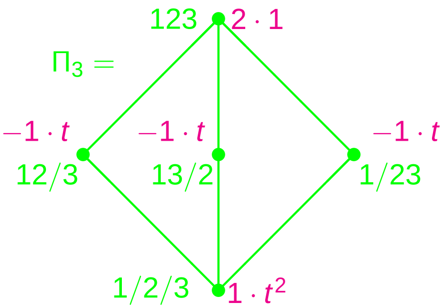
$\Pi_3 =$



In general $q(\Pi_n; t) = (t-1)(t-2)\cdots(t-n+1)$. Not clear!

Example.

$$q(\Pi_3) = t^2 - 3t + 2$$
$$= (t-1)(t-2)$$



In general $q(\Pi_n; t) = (t-1)(t-2)\cdots(t-n+1)$. Not clear!

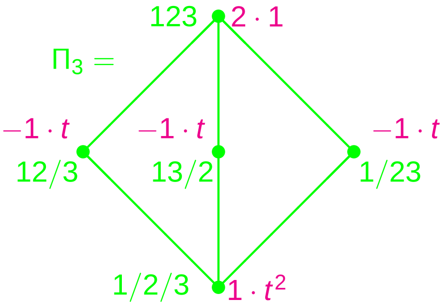
We would like to have a technique which would

1. prove the formula for $q(\Pi_n; t)$,

Example.

$$q(\Pi_3) = t^2 - 3t + 2$$
$$= (t-1)(t-2)$$

$\Pi_3 =$



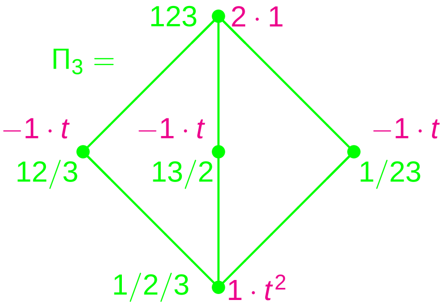
In general $q(\Pi_n; t) = (t-1)(t-2)\cdots(t-n+1)$. Not clear!

We would like to have a technique which would

1. prove the formula for $q(\Pi_n; t)$,
2. explain why these $q(P; t)$ factor over $\mathbb{Z}_{\geq 0}$.

Example.

$$q(\Pi_3) = t^2 - 3t + 2$$
$$= (t-1)(t-2)$$



In general $q(\Pi_n; t) = (t-1)(t-2)\cdots(t-n+1)$. Not clear!

We would like to have a technique which would

1. prove the formula for $q(\Pi_n; t)$,
2. explain why these $q(P; t)$ factor over $\mathbb{Z}_{\geq 0}$.

We will use a technique based on graph theory. Two other techniques (one using the theory of hyperplane arrangements and one using properties of posets) are given in the paper.

Outline

The Characteristic Polynomial

The Chromatic Polynomial

The Bond Lattice

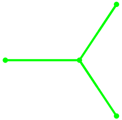
The Connection

Let G be a graph with vertices V and edges E.

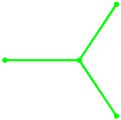
Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \longrightarrow C$.

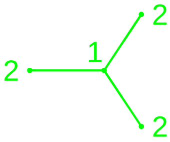
Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \longrightarrow C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

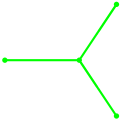
Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

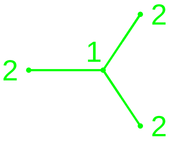
Example. If $G =$  and $C = \{1, 2\}$ then



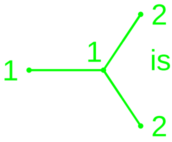
is a proper coloring of G

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \longrightarrow C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

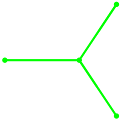


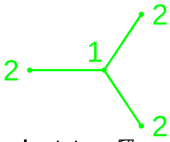
is a proper coloring of G while

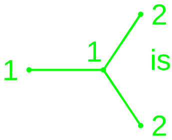


is not.

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

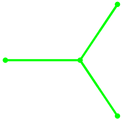


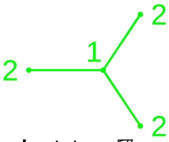
is a proper coloring of G while 

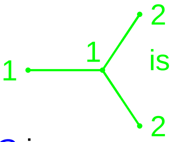
is not.

Let $t \in \mathbb{Z}_{\geq 0}$.

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



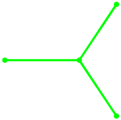
is a proper coloring of G while


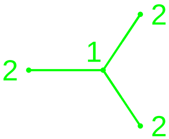
is not.

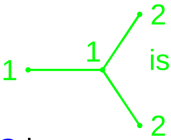
Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

$$p(G) = p(G; t) = \# \text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

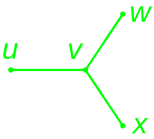


is a proper coloring of G while


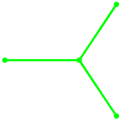
is not.

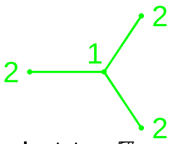
Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

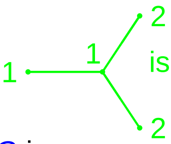
$$p(G) = p(G; t) = \# \text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

Example. If $G =$ 

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

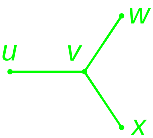


is a proper coloring of G while 

is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

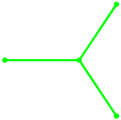
$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

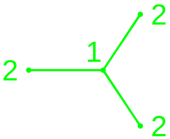
Example. If $G =$ 

then

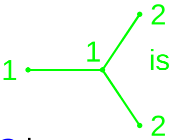
$p(G; t) \quad = \quad$ # of ways to color u, then v, then w, then x

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



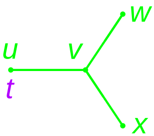
is a proper coloring of G while



is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

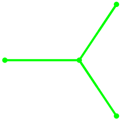
$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

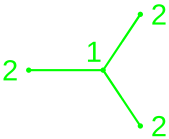
Example. If $G =$ 

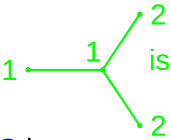
then

$$\begin{aligned} p(G;t) &= \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\ &= t \end{aligned}$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then

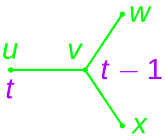


is a proper coloring of G while 

is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

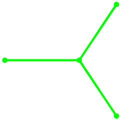
$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

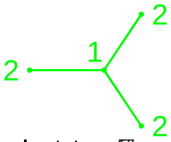
Example. If $G =$ 

then

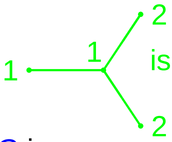
$$
\begin{aligned}
p(G; t) &= \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\
&= t(t-1)
\end{aligned}
$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



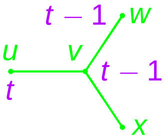
is a proper coloring of G while



is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

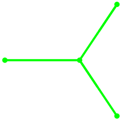
$$p(G) = p(G;t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

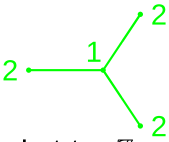
Example. If $G =$ 

then

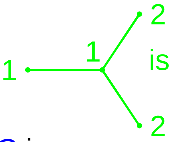
$$\begin{aligned} p(G;t) &= \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\ &= t(t-1)(t-1) \end{aligned}$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



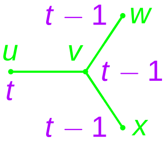
is a proper coloring of G while



is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

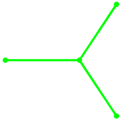
$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

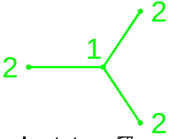
Example. If $G =$ 

then

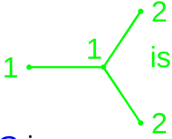
$$\begin{aligned} p(G; t) &= \#\text{ of ways to color } u, \text{ then } v, \text{ then } w, \text{ then } x \\ &= t(t-1)(t-1)(t-1) \end{aligned}$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



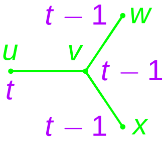
is a proper coloring of G while



is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

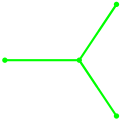
$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

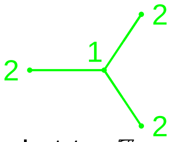
Example. If $G =$ 

then

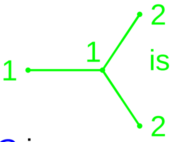
$$\begin{aligned} p(G; t) &= \#\text{ of ways to color } u, \text{ then } v, \text{ then } w, \text{ then } x \\ &= t(t-1)(t-1)(t-1) = t(t-1)^3 \end{aligned}$$

Let G be a graph with vertices V and edges E. Given a set C, called the *color set*, a *coloring of G* is a function $\kappa : V \to C$. A coloring is *proper* if for each edge $uv \in E$ we have $\kappa(u) \neq \kappa(v)$.

Example. If $G =$  and $C = \{1, 2\}$ then



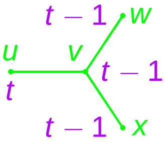
is a proper coloring of G while



is not.

Let $t \in \mathbb{Z}_{\geq 0}$. The *chromatic polynomial of G* is

$$p(G) = p(G; t) = \#\text{ of proper } \kappa : V \to \{1, \ldots, t\}.$$

Example. If $G =$ 

then

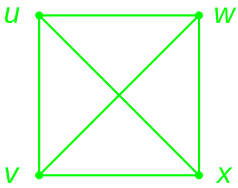
$$\begin{aligned} p(G;t) &= \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\ &= t(t-1)(t-1)(t-1) = t(t-1)^3 \end{aligned}$$

In general of any tree T with $|E| = n$: $\quad p(G;t) = t(t-1)^n$.

Let K_V be the complete graph having all possible edges between elements of V.

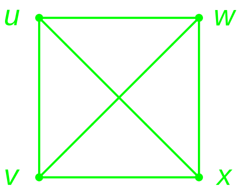
Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Example. If $G = K_4 =$ 

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Example. If $G = K_4 =$

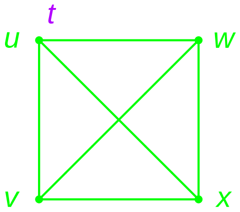


then

$$p(K_4; t) \quad = \quad \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Example. If $G = K_4 =$

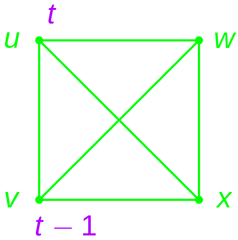


then

$$p(K_4; t) = \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$
$$= t$$

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

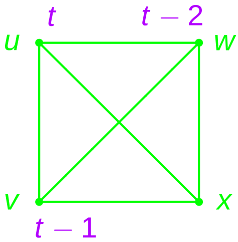
Example. If $G = K_4 =$



then

$$p(K_4; t) = \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$
$$= t(t-1)$$

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

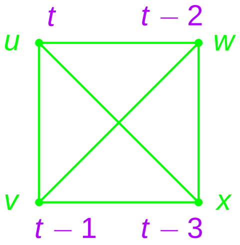
Example. If $G = K_4 =$ 

then

$$p(K_4; t) = \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$
$$= t(t-1)(t-2)$$

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

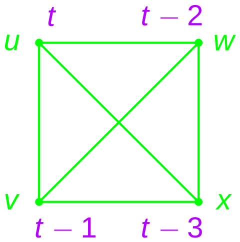
Example. If $G = K_4 =$



then

$$p(K_4; t) = \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$
$$= t(t-1)(t-2)(t-3)$$

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

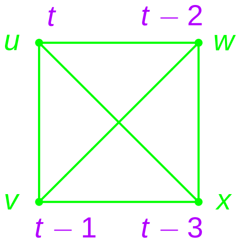
Example. If $G = K_4 =$  then

$$\begin{aligned} p(K_4; t) &= \#\text{ of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\ &= t(t-1)(t-2)(t-3) \end{aligned}$$

In general $p(K_n; t) = t(t-1)\cdots(t-n+1)$.

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Example. If $G = K_4 =$



then

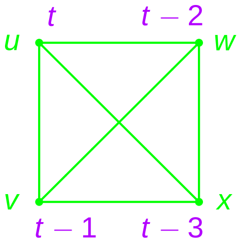
$$p(K_4; t) = \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x$$
$$= t(t-1)(t-2)(t-3)$$

In general $p(K_n; t) = t(t-1)\cdots(t-n+1)$.

We need to explain

1. why does $p(G; t)$ always seem to be a polynomial in t?

Let K_V be the complete graph having all possible edges between elements of V. If $|V| = n$ then we also write K_n for K_V.

Example. If $G = K_4 =$  then

$$
\begin{aligned}
p(K_4; t) &= \text{\# of ways to color } u\text{, then } v\text{, then } w\text{, then } x \\
&= t(t-1)(t-2)(t-3)
\end{aligned}
$$

In general $p(K_n; t) = t(t-1)\cdots(t-n+1)$.

We need to explain

1. why does $p(G; t)$ always seem to be a polynomial in t?
2. why do $p(T; t)$ where T is a tree and $p(K_n; t)$ seem to be related to $q(B_n; t)$ and $q(\Pi_n; t)$, respectively?

Outline

The Characteristic Polynomial

The Chromatic Polynomial

The Bond Lattice

The Connection

Let $k(G)$ denote the number of components of G.

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G.

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

1. $V(H) = V(G)$,

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

1. $V(H) = V(G)$, i.e., H is *spanning*.

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

1. $V(H) = V(G)$, i.e., H is *spanning*.
2. $u \sim_H v$ and $uv \in E(G)$ implies $uv \in E(H)$,

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

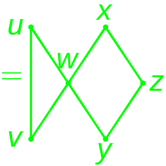
A subgraph $H \subseteq G$ is a *bond* if

1. $V(H) = V(G)$, i.e., H is *spanning*.
2. $u \sim_H v$ and $uv \in E(G)$ implies $uv \in E(H)$, i.e., each component of H is an *induced subgraph*.

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

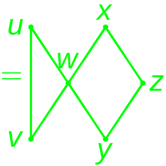
1. $V(H) = V(G)$, i.e., H is *spanning*.
2. $u \sim_H v$ and $uv \in E(G)$ implies $uv \in E(H)$, i.e., each component of H is an *induced subgraph*.

Example. If $G =$ 

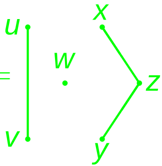
Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

1. $V(H) = V(G)$, i.e., H is *spanning*.
2. $u \sim_H v$ and $uv \in E(G)$ implies $uv \in E(H)$, i.e., each component of H is an *induced subgraph*.

Example. If $G =$ 

then

$H =$ 

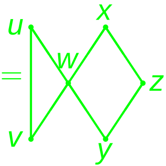
is a bond of G,

Let $k(G)$ denote the number of components of G. We also write $u \sim v$ if u and v are in the same component of G. If we wish to be specific about the graph in question, we will use notation such as $V(G)$ or $\sim_G$.

A subgraph $H \subseteq G$ is a *bond* if

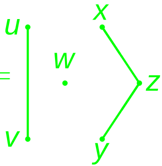
1. $V(H) = V(G)$, i.e., H is *spanning*.
2. $u \sim_H v$ and $uv \in E(G)$ implies $uv \in E(H)$, i.e., each component of H is an *induced subgraph*.

Example. If $G =$

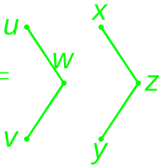


then

$H =$



is a bond of G, while $H =$



is not since $u \sim_H v$ and $uv \in E(G)$ but $uv \notin E(H)$.

The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

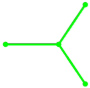
partially ordered by inclusion.

The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

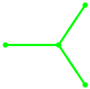
$T =$ 

The *bond lattice of G* is

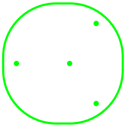
$$L(G) = \{H \;:\; H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

$T =$ 

$L(T) =$



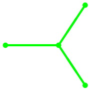
The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

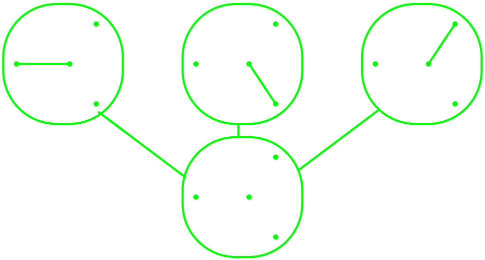
partially ordered by inclusion.

Example.

$T =$



$L(T) =$

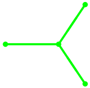


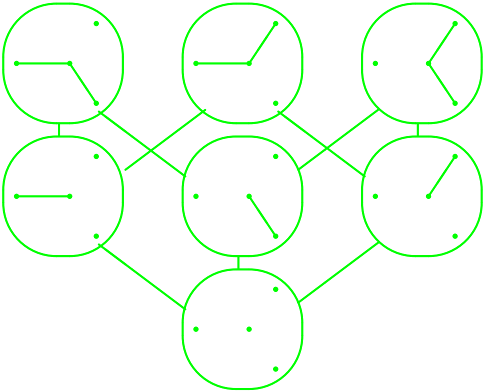
The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

$T =$ 

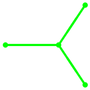
$L(T) =$ 

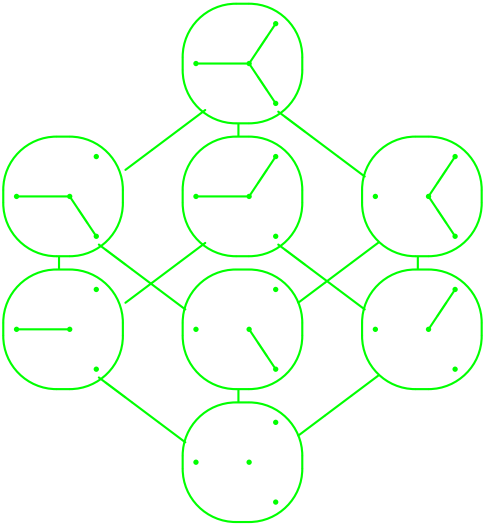
The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

$T =$ 

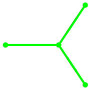
$L(T) =$ 

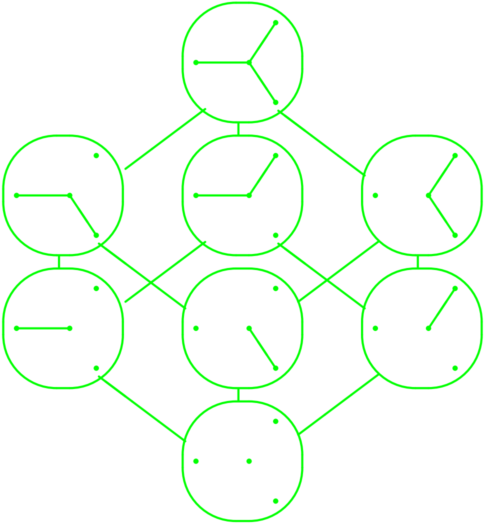
The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

$T =$ 

$L(T) =$ 

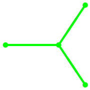
So $L(T) \cong B_3$.

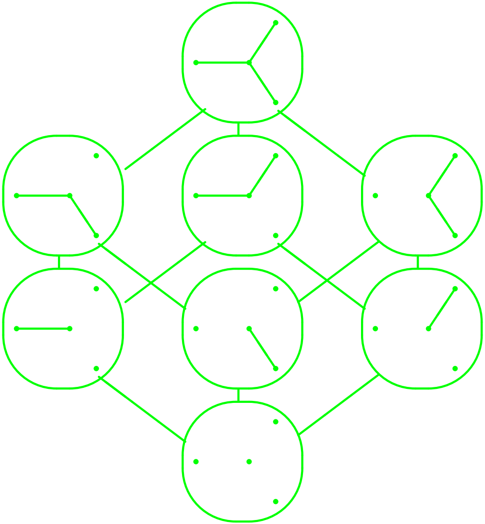
The *bond lattice of G* is

$$L(G) = \{H \ : \ H \text{ is a bond of } G\}$$

partially ordered by inclusion.

Example.

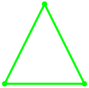
$T =$ 

$L(T) =$ 

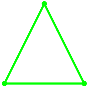
So $L(T) \cong B_3$.

In general, if T is a tree with n edges then $L(T) \cong B_n$.

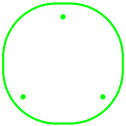
Example.

$K_3 =$ 

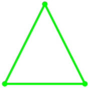
Example.

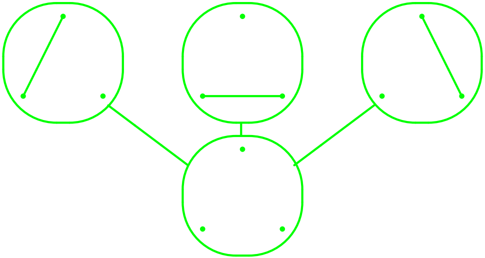
$K_3 =$ 

$L(K_3) =$

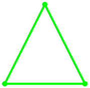


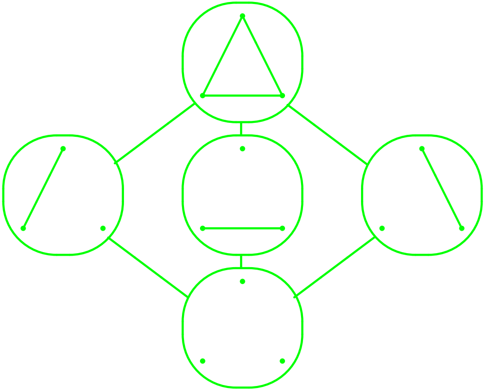
Example.

$K_3 =$ 

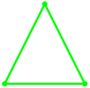
$L(K_3) =$ 

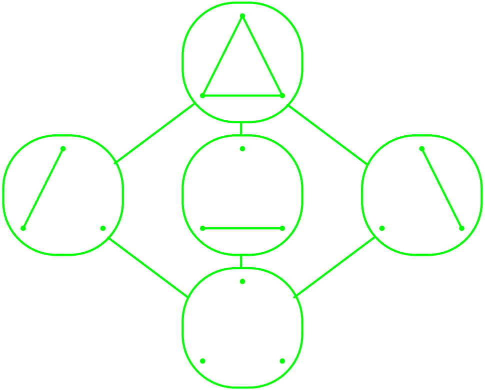
Example.

$K_3 =$ 

$L(K_3) =$ 

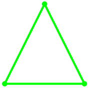
Example.

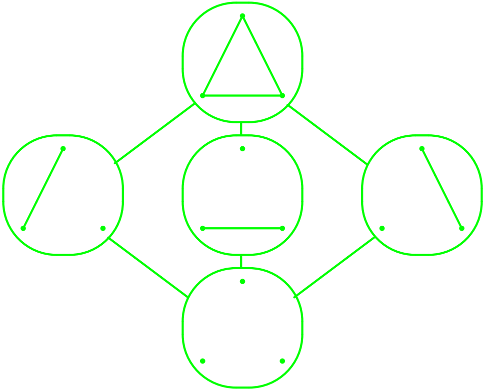
$K_3 =$ 

$L(K_3) =$ 

So $L(K_3) \cong \Pi_3$.

Example.

$K_3 =$ 

$L(K_3) =$ 

So $L(K_3) \cong \Pi_3$.

In general, $L(K_n) \cong \Pi_n$.

Outline

The Characteristic Polynomial

The Chromatic Polynomial

The Bond Lattice

The Connection

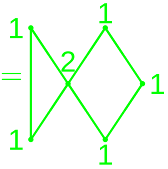
Given a coloring $\kappa : V \to \{1, \dots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

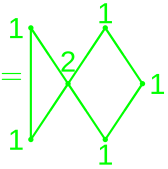
$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

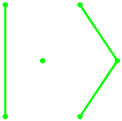
Example. If $\kappa =$



Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

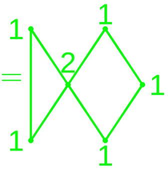
$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

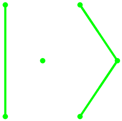
Example. If $\kappa =$ 

then $H_\kappa =$ 

Given a coloring $\kappa : V \rightarrow \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

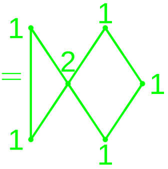
Lemma

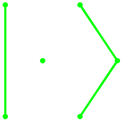
For any graph G

(a) *H_κ is a bond for any κ.*

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

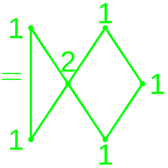
For any graph G

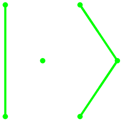
(a) *H_κ is a bond for any κ.*

(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

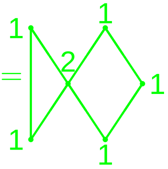
(a) *H_κ is a bond for any κ.*

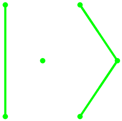
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a).

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

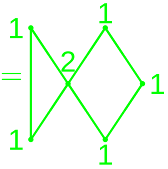
(a) *H_κ is a bond for any κ.*

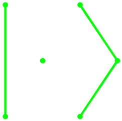
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition.

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

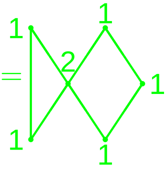
(a) *H_κ is a bond for any κ.*

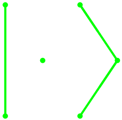
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$.

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

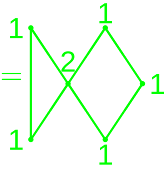
(a) *H_κ is a bond for any κ.*

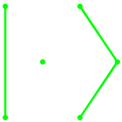
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ.

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

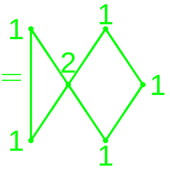
(a) *H_κ is a bond for any κ.*

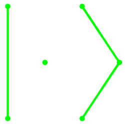
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ. So $\kappa(u) = \kappa(v)$ since every pair of adjacent vertices of W have the same color.

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

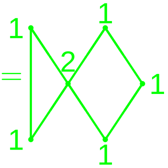
(a) *H_κ is a bond for any κ.*

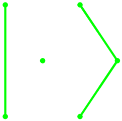
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ. So $\kappa(u) = \kappa(v)$ since every pair of adjacent vertices of W have the same color. Thus $uv \in E(H_\kappa)$ as desired.

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

(a) *H_κ is a bond for any κ.*

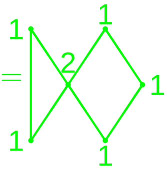
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

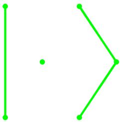
Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ. So $\kappa(u) = \kappa(v)$ since every pair of adjacent vertices of W have the same color. Thus $uv \in E(H_\kappa)$ as desired.

Proof of (b). κ is proper iff $\kappa(u) \neq \kappa(v)$ for all $uv \in E(G)$

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of κ* is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

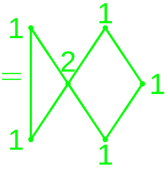
(a) *H_κ is a bond for any κ.*

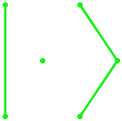
(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ. So $\kappa(u) = \kappa(v)$ since every pair of adjacent vertices of W have the same color. Thus $uv \in E(H_\kappa)$ as desired.
Proof of (b). κ is proper iff $\kappa(u) \neq \kappa(v)$ for all $uv \in E(G)$ iff $E(H_\kappa) = \emptyset$

Given a coloring $\kappa : V \to \{1, \ldots, t\}$ of G, the *kernel of* κ is the spanning subgraph $H_\kappa \subseteq G$ where

$$uv \in E(H_\kappa) \text{ if and only if } uv \in E(G) \text{ and } \kappa(u) = \kappa(v).$$

Example. If $\kappa =$ 

then $H_\kappa =$ 

Lemma

For any graph G

(a) *H_κ is a bond for any κ.*

(b) *κ is proper iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$.*

Proof of (a). H_κ is spanning by definition. Now suppose $u \sim_{H_\kappa} v$ and $uv \in E(G)$. Since $u \sim_{H_\kappa} v$ there is a u–v walk W in H_κ. So $\kappa(u) = \kappa(v)$ since every pair of adjacent vertices of W have the same color. Thus $uv \in E(H_\kappa)$ as desired.

Proof of (b). κ is proper iff $\kappa(u) \neq \kappa(v)$ for all $uv \in E(G)$ iff $E(H_\kappa) = \emptyset$ iff $H_\kappa = \hat{0}_{L(G)}$. □

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) $L(G)$ *is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) $L(G)$ *is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b).

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\quad \mathrm{rk}\, H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\mathrm{rk}\, H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \qquad (1)$$

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\mathrm{rk}\, H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \qquad (1)$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \tag{1}$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G; t)$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) $L(G)$ *is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \tag{1}$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G;t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with $\operatorname{rk} H = n - k(H)$.*

(b) *$p(G; t) = t^{k(G)} q(L(G); t)$.*

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \qquad (1)$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G; t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'. So $f(H') = t^{k(H')}$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \qquad (1)$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G;t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'. So $f(H') = t^{k(H')}$. Thus (1) becomes

$$p(G;t) = \sum_{H' \in L(G)} \mu(H') t^{k(H')}. \qquad (2)$$

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \tag{1}$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G;t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'. So $f(H') = t^{k(H')}$. Thus (1) becomes

$$p(G;t) = \sum_{H' \in L(G)} \mu(H') t^{k(H')}. \tag{2}$$

By (a), $k(H') + \operatorname{rk} H' = n = k(G) + \operatorname{rk} G$.

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G; t) = t^{k(G)} q(L(G); t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \tag{1}$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G; t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'. So $f(H') = t^{k(H')}$. Thus (1) becomes

$$p(G; t) = \sum_{H' \in L(G)} \mu(H') t^{k(H')}. \tag{2}$$

By (a), $k(H') + \operatorname{rk} H' = n = k(G) + \operatorname{rk} G$. Plugging into (2) gives

$$p(G; t) = \sum_{H' \in L(G)} \mu(H') t^{k(G) + \operatorname{rk} G - \operatorname{rk} H'}$$

Theorem (Rota, 1964)

Let G be a graph with $|V| = n$.

(a) *$L(G)$ is a graded lattice with* $\operatorname{rk} H = n - k(H)$.

(b) $p(G;t) = t^{k(G)} q(L(G);t)$.

Proof of (b). Define $f, g : L(G) \to \mathbb{Z}$ by

1. $f(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa \supseteq H$,
2. $g(H) = \#$ of $\kappa : V(G) \to \{1, \ldots, t\}$ with $H_\kappa = H$,

Since H_κ is always a bond, $f(H) = \sum_{H' \supseteq H} g(H')$. By the MIT

$$g(\hat{0}) = \sum_{H' \supseteq \hat{0}} \mu(H') f(H'). \tag{1}$$

But $g(\hat{0})$ counts κ with $H_\kappa = \hat{0}$ so, by the lemma, $g(\hat{0}) = p(G;t)$. Also, given H' then $H_\kappa \supseteq H'$ iff κ is constant on the components of H'. So $f(H') = t^{k(H')}$. Thus (1) becomes

$$p(G;t) = \sum_{H' \in L(G)} \mu(H') t^{k(H')}. \tag{2}$$

By (a), $k(H') + \operatorname{rk} H' = n = k(G) + \operatorname{rk} G$. Plugging into (2) gives

$$p(G;t) = \sum_{H' \in L(G)} \mu(H') t^{k(G) + \operatorname{rk} G - \operatorname{rk} H'} = t^{k(G)} q(L(G);t) \qquad \square$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$q(B_n;t) \quad = \quad t^{-k(T)} p(T;t)$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \end{aligned}$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \\ &= (t-1)^n. \end{aligned}$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \\ &= (t-1)^n. \end{aligned}$$

For Π_n we have $L(K_n) \cong \Pi_n$.

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \\ &= (t-1)^n. \end{aligned}$$

For Π_n we have $L(K_n) \cong \Pi_n$. So by Rota's Theorem

$$q(\Pi_n;t) = t^{-k(K_n)} p(K_n;t)$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \\ &= (t-1)^n. \end{aligned}$$

For Π_n we have $L(K_n) \cong \Pi_n$. So by Rota's Theorem

$$\begin{aligned} q(\Pi_n;t) &= t^{-k(K_n)} p(K_n;t) \\ &= t^{-1} \cdot t(t-1)\cdots(t-n+1) \end{aligned}$$

Rota: $p(G;t) = t^{k(G)} q(L(G);t)$.

Corollary

1. $q(B_n;t) = (t-1)^n$.
2. $q(\Pi_n;t) = (t-1)(t-2)\cdots(t-n+1)$.

Proof. For B_n we have $L(T) \cong B_n$ for any tree T with n edges. So by Rota's Theorem

$$\begin{aligned} q(B_n;t) &= t^{-k(T)} p(T;t) \\ &= t^{-1} \cdot t(t-1)^n \\ &= (t-1)^n. \end{aligned}$$

For Π_n we have $L(K_n) \cong \Pi_n$. So by Rota's Theorem

$$\begin{aligned} q(\Pi_n;t) &= t^{-k(K_n)} p(K_n;t) \\ &= t^{-1} \cdot t(t-1)\cdots(t-n+1) \\ &= (t-1)\cdots(t-n+1). \qquad \square \end{aligned}$$