Möbius Functions of Posets I: Introduction to Partially Ordered Sets

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

June 25, 2007

Outline

Example A: Combinatorics.

Example A: Combinatorics.

Given a set, S, let

$$\#S = |S| = \text{cardinality of } S.$$

Example A: Combinatorics.
Given a set, S, let

$$\#S = |S| = \text{cardinality of } S.$$

The Principle of Inclusion-Exclusion or PIE is a very useful tool in enumerative combinatorics.

Example A: Combinatorics.
Given a set, S, let

$$\#S = |S| = \text{cardinality of } S.$$

The Principle of Inclusion-Exclusion or PIE is a very useful tool in enumerative combinatorics.

Theorem (PIE)

Let U be a finite set and $U_1, \ldots, U_n \subseteq U$.

$$|U - \bigcup_{i=1}^{n} U_i| = |U| - \sum_{1\le i\le n} |U_i| + \sum_{1\le i<j\le n} |U_i \cap U_j| - \cdots + (-1)^n |\bigcap_{i=1}^{n} U_i|. \qquad \square$$

Example B: Theory of Finite Differences.

Example B: Theory of Finite Differences.

Let

$$\mathbb{Z}_{\geq 0} = \text{the nonnegative integers.}$$

Example B: Theory of Finite Differences.

Let

$$\mathbb{Z}_{\geq 0} = \text{the nonnegative integers.}$$

If one takes a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ then there is an analogue of the derivative, namely the difference operator

$$\Delta f(n) = f(n) - f(n-1)$$

(where $f(-1) = 0$ by definition).

Example B: Theory of Finite Differences.

Let

$$\mathbb{Z}_{\geq 0} = \text{the nonnegative integers.}$$

If one takes a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ then there is an analogue of the derivative, namely the difference operator

$$\Delta f(n) = f(n) - f(n-1)$$

(where $f(-1) = 0$ by definition). There is also an analogue of the integral, namely the summation operator

$$Sf(n) = \sum_{i=0}^{n} f(i).$$

Example B: Theory of Finite Differences.

Let

$$\mathbb{Z}_{\geq 0} = \text{the nonnegative integers.}$$

If one takes a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ then there is an analogue of the derivative, namely the difference operator

$$\Delta f(n) = f(n) - f(n-1)$$

(where $f(-1) = 0$ by definition). There is also an analogue of the integral, namely the summation operator

$$Sf(n) = \sum_{i=0}^{n} f(i).$$

The Fundamental Theorem of the Difference Calculus or FTDC is as follows.

Theorem (FTDC)

If $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ then

$$\Delta Sf(n) = f(n). \qquad \square$$

Example C: Number Theory

Example C: Number Theory

If $d, n \in \mathbb{Z}$ then write $d|n$ if d divides evenly into n.

Example C: Number Theory

If $d, n \in \mathbb{Z}$ then write $d|n$ if d divides evenly into n. The number-theoretic Möbius function is $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free,} \\ (-1)^k & \text{if } n = \text{product of } k \text{ distinct primes.} \end{cases}$$

Example C: Number Theory

If $d, n \in \mathbb{Z}$ then write $d|n$ if d divides evenly into n. The number-theoretic Möbius function is $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free,} \\ (-1)^k & \text{if } n = \text{product of } k \text{ distinct primes.} \end{cases}$$

The importance of μ lies in the number-theoretic Möbius Inversion Theorem or MIT.

Example C: Number Theory

If $d, n \in \mathbb{Z}$ then write $d|n$ if d divides evenly into n. The number-theoretic Möbius function is $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free,} \\ (-1)^k & \text{if } n = \text{product of } k \text{ distinct primes.} \end{cases}$$

The importance of μ lies in the number-theoretic Möbius Inversion Theorem or MIT.

Theorem (Number Theory MIT)

Let $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$ satisfy

$$f(n) = \sum_{d|n} g(d)$$

for all $n \in \mathbb{Z}_{>0}$. Then

$$g(n) = \sum_{d|n} \mu(n/d) f(d). \qquad \square$$

Möbius inversion over partially ordered sets is important for the following reasons.

Möbius inversion over partially ordered sets is important for the following reasons.

1. It unifies and generalizes the three previous examples.

Möbius inversion over partially ordered sets is important for the following reasons.

1. It unifies and generalizes the three previous examples.
2. It makes the number-theoretic definition transparent.

Möbius inversion over partially ordered sets is important for the following reasons.

1. It unifies and generalizes the three previous examples.
2. It makes the number-theoretic definition transparent.
3. It encodes topological information about partially ordered sets.

Möbius inversion over partially ordered sets is important for the following reasons.

1. It unifies and generalizes the three previous examples.
2. It makes the number-theoretic definition transparent.
3. It encodes topological information about partially ordered sets.
4. It can be used to solve combinatorial problems.

Outline

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Given any poset notation, if we wish to be specific about the poset P involved, we attach P as a subscript. For example, using $\leq_P$ for $\leq$.

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Given any poset notation, if we wish to be specific about the poset P involved, we attach P as a subscript. For example, using $\leq_P$ for $\leq$. We also adopt the usual conventions for inequalities. For example, $x < y$ means $x \leq y$ and $x \neq y$.

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Given any poset notation, if we wish to be specific about the poset P involved, we attach P as a subscript. For example, using $\leq_P$ for $\leq$. We also adopt the usual conventions for inequalities. For example, $x < y$ means $x \leq y$ and $x \neq y$.

If $x, y \in P$ then *x is covered by y* or *y covers x*, written $x \lhd y$, if $x < y$ and there is no z with $x < z < y$.

A *partially ordered set* or *poset* is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$:

1. (reflexivity) $x \leq x$,
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Given any poset notation, if we wish to be specific about the poset P involved, we attach P as a subscript. For example, using $\leq_P$ for $\leq$. We also adopt the usual conventions for inequalities. For example, $x < y$ means $x \leq y$ and $x \neq y$.
If $x, y \in P$ then *x is covered by y* or *y covers x*, written $x \lhd y$, if $x < y$ and there is no z with $x < z < y$. The *Hasse diagram* of P is the (directed) graph with vertices P and an edge from x up to y if $x \lhd y$.

Example: The Chain.

Example: The Chain.

The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.

Example: The Chain.

The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.

C_3=

Example: The Chain.
The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.

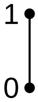
C_3=

0 •

Example: The Chain.

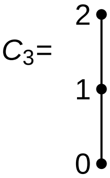
The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.

C_3=



Example: The Chain.

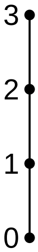
The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.



Example: The Chain.

The *chain of length n* is $C_n = \{0, 1, \ldots, n\}$ with the usual $\leq$ on the integers.

$C_3 =$



Example: The Boolean Algebra.

Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \dots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$$B_3 =$$

Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

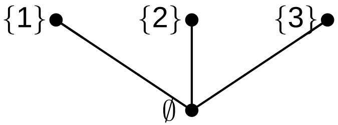
$B_3 =$

Example: The Boolean Algebra.
The *Boolean algebra* is

$$B_n = \{S \; : \; S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$B_3 =$



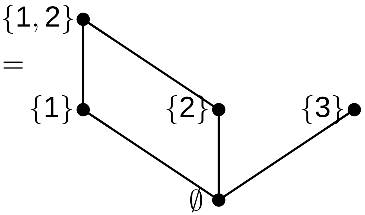
Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$B_3 =$



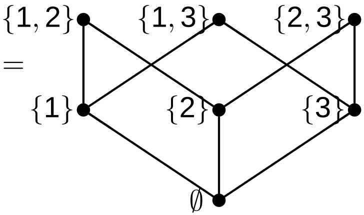
Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$B_3 =$



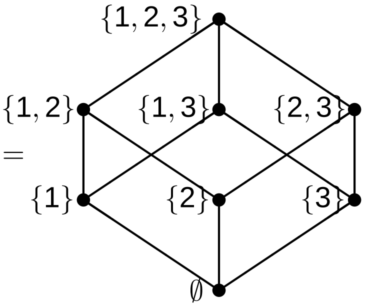
Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \leq T$ if and only if $S \subseteq T$.

$B_3 =$



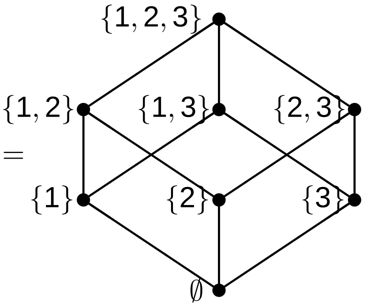
Example: The Boolean Algebra.

The *Boolean algebra* is

$$B_n = \{S \ : \ S \subseteq \{1, 2, \ldots, n\}\}$$

partially ordered by $S \le T$ if and only if $S \subseteq T$.

$B_3 =$



Note that B_3 looks like a cube.

Example: The Divisor Lattice.

Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

$$D_{18} =$$

Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

$$D_{18} =$$

1 •

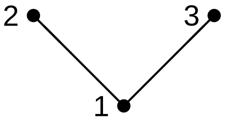
Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

$D_{18} =$



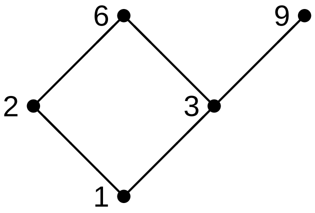
Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

$D_{18} =$

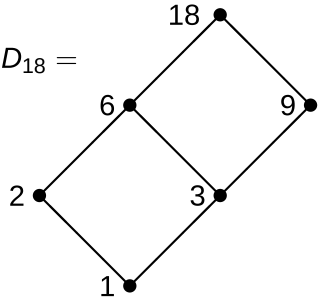


Example: The Divisor Lattice.

Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

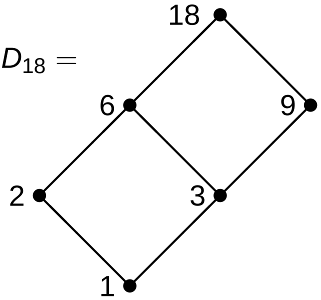
$D_{18} =$ 

Example: The Divisor Lattice.
Given $n \in \mathbb{Z}_{>0}$ the corresponding *divisor lattice* is

$$D_n = \{d \in \mathbb{Z}_{>0} \ : \ d|n\}$$

partially ordered by $c \leq_{D_n} d$ if and only if $c|d$.

$D_{18} =$

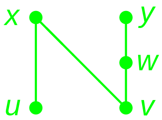


Note that D_{18} looks like a rectangle.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$.

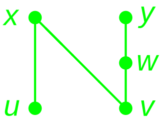
In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.



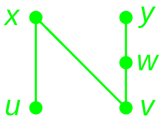
Example. The poset on the left has minimal elements u and v,

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.



Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

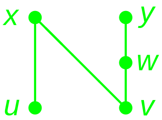
In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.



Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

A poset *has a zero* if it has a unique minimal element, $\hat{0}$.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

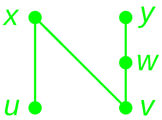


Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

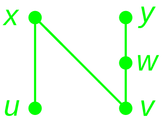
Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

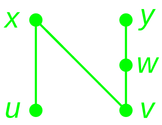


A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



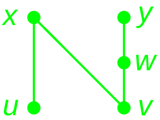
A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0,$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



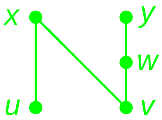
A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0, \ \hat{1}_{C_n} = n,$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

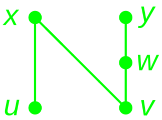


A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.



Example. The poset on the left has minimal elements u and v, and maximal elements x and y.

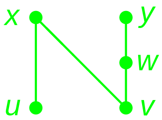
A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \dots, n\},$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



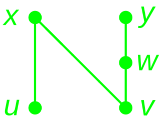
A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \ldots, n\},\ \hat{0}_{D_n} = 1,$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



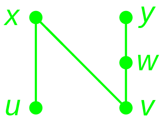
A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \ldots, n\},\ \hat{0}_{D_n} = 1,\ \hat{1}_{D_n} = n.$$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

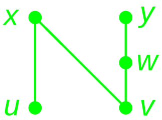
$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \ldots, n\},\ \hat{0}_{D_n} = 1,\ \hat{1}_{D_n} = n.$

If $x \le y$ in P then the corresponding *closed interval* is

$$[x, y] = \{z \ : \ x \le z \le y\}.$$

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \ldots, n\},\ \hat{0}_{D_n} = 1,\ \hat{1}_{D_n} = n.$$

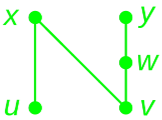
If $x \le y$ in P then the corresponding *closed interval* is

$$[x, y] = \{z : x \le z \le y\}.$$

Open and half-open intervals are defined analogously.

In a poset P, a *minimal* element is $x \in P$ such that there is no $y \in P$ with $y < x$. A *maximal* element is $x \in P$ such that there is no $y \in P$ with $y > x$.

Example. The poset on the left has minimal elements u and v, and maximal elements x and y.



A poset *has a zero* if it has a unique minimal element, $\hat{0}$. A poset *has a one* if it has a unique maximal element, $\hat{1}$. A poset if *bounded* if it has both a $\hat{0}$ and a $\hat{1}$.

Example. Our three fundamental examples are bounded:

$$\hat{0}_{C_n} = 0,\ \hat{1}_{C_n} = n,\ \hat{0}_{B_n} = \emptyset,\ \hat{1}_{B_n} = \{1, \ldots, n\},\ \hat{0}_{D_n} = 1,\ \hat{1}_{D_n} = n.$$

If $x \leq y$ in P then the corresponding *closed interval* is

$$[x, y] = \{z \ : \ x \leq z \leq y\}.$$

Open and half-open intervals are defined analogously. Note that $[x, y]$ is a poset in its own right and it has a zero and a one:

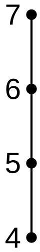
$$\hat{0}_{[x,y]} = x, \qquad \hat{1}_{[x,y]} = y.$$

Example: The Chain.

In C_9 we have the interval $[4, 7]$:

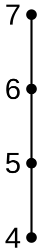
Example: The Chain.

In C_9 we have the interval $[4, 7]$:



Example: The Chain.

In C_9 we have the interval $[4, 7]$:



This interval looks like C_3.

Example: The Boolean Algebra.

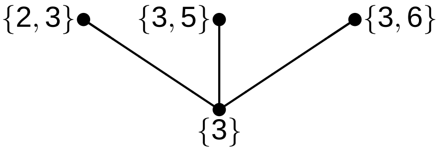
In B_7 we have the interval $[\{3\}, \{2, 3, 5, 6\}]$:

Example: The Boolean Algebra.

In B_7 we have the interval $[\{3\}, \{2,3,5,6\}]$:

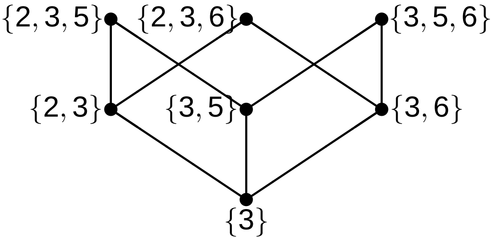
Example: The Boolean Algebra.

In B_7 we have the interval $[\{3\}, \{2,3,5,6\}]$:



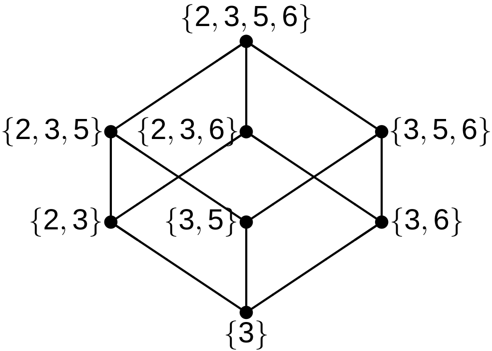
Example: The Boolean Algebra.

In B_7 we have the interval $[\{3\}, \{2,3,5,6\}]$:



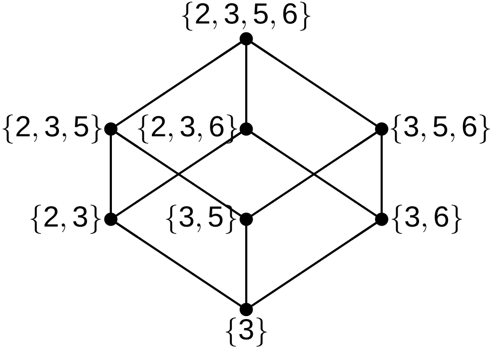
Example: The Boolean Algebra.

In B_7 we have the interval $[\{3\}, \{2,3,5,6\}]$:



Example: The Boolean Algebra.

In B_7 we have the interval $[\{3\}, \{2, 3, 5, 6\}]$:



Note that this interval looks like B_3.

Example: The Divisor Lattice.

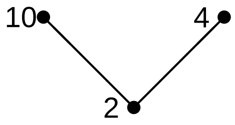
In D_{80} we have the interval $[2, 40]$:

Example: The Divisor Lattice.

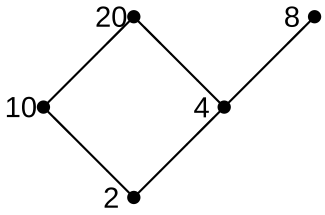
In D_{80} we have the interval $[2, 40]$:

Example: The Divisor Lattice.

In D_{80} we have the interval $[2, 40]$:

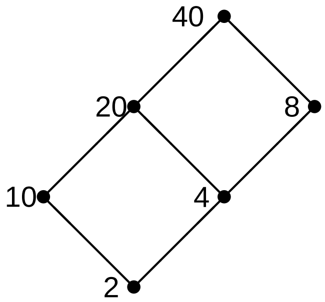


Example: The Divisor Lattice.
In D_{80} we have the interval $[2, 40]$:



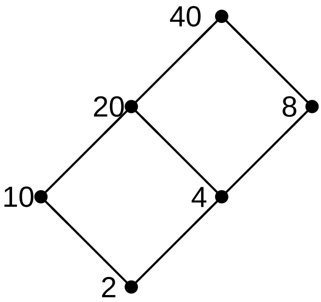
Example: The Divisor Lattice.

In D_{80} we have the interval $[2, 40]$:



Example: The Divisor Lattice.

In D_{80} we have the interval $[2, 40]$:



Note that this interval looks like D_{18}.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

We say P is a *lattice* if every $x, y \in P$ have both a meet and a join.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

We say P is a *lattice* if every $x, y \in P$ have both a meet and a join.

Example.

1. C_n is a lattice with $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

We say P is a *lattice* if every $x, y \in P$ have both a meet and a join.

Example.

1. C_n is a lattice with $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.
2. B_n is a lattice with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

We say P is a *lattice* if every $x, y \in P$ have both a meet and a join.

Example.

1. C_n is a lattice with $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.
2. B_n is a lattice with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.
3. D_n is a lattice with $c \wedge d = \gcd\{c, d\}$ and $c \vee d = \mathrm{lcm}\{c, d\}$.

Outline

Motivating Examples

Poset Basics

Isomorphism and Products

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \quad \Longrightarrow \quad f(x) \leq_Q f(y).$$

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \quad \Longrightarrow \quad f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \quad \Longrightarrow \quad f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c, d] \cong D_{d/c}$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c, d] \cong D_{d/c}$.

Proof for C_n. Define $f : [i, j] \to C_{j-i}$ by $f(k) = k - i$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i,j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S,T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c,d] \cong D_{d/c}$.

Proof for C_n. Define $f : [i,j] \to C_{j-i}$ by $f(k) = k - i$. Then f is order preserving since

$$k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l).$$

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \quad \Longrightarrow \quad f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i,j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S,T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c,d] \cong D_{d/c}$.

Proof for C_n. Define $f : [i,j] \to C_{j-i}$ by $f(k) = k - i$. Then f is order preserving since

$$k \leq l \quad \Longrightarrow \quad k - i \leq l - i \quad \Longrightarrow \quad f(k) \leq f(l).$$

Also f is bijective with inverse $f^{-1}(k) = k + i$.

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c, d] \cong D_{d/c}$.

Proof for C_n. Define $f : [i, j] \to C_{j-i}$ by $f(k) = k - i$. Then f is order preserving since

$$k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l).$$

Also f is bijective with inverse $f^{-1}(k) = k + i$. It is easy to check that f^{-1} is order preserving. □

For posets P and Q, an *order preserving map* is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An *isomorphism* is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving. In this case P and Q are *isomorphic*, written $P \cong Q$.

Proposition

If $i \leq j$ in C_n then $[i, j] \cong C_{j-i}$.
If $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.
If $c|d$ in D_n then $[c, d] \cong D_{d/c}$.

Proof for C_n. Define $f : [i, j] \to C_{j-i}$ by $f(k) = k - i$. Then f is order preserving since

$$k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l).$$

Also f is bijective with inverse $f^{-1}(k) = k + i$. It is easy to check that f^{-1} is order preserving. □

Exercise. Prove the other two parts of the Proposition.

If P and Q are posets, then their *product* is

$$P \times Q = \{(a,x) \ : a \in P,\ x \in Q\}$$

partially ordered by

$$(a,x) \leq_{P\times Q} (b,y) \quad \Longleftrightarrow \quad a \leq_P b \quad \text{and} \quad x \leq_Q y.$$

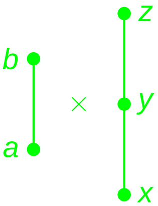
If P and Q are posets, then their *product* is

$$P \times Q = \{(a, x)\ : a \in P,\ x \in Q\}$$

partially ordered by

$$(a, x) \leq_{P\times Q} (b, y) \quad \Longleftrightarrow \quad a \leq_P b \quad \text{and} \quad x \leq_Q y.$$

Example.



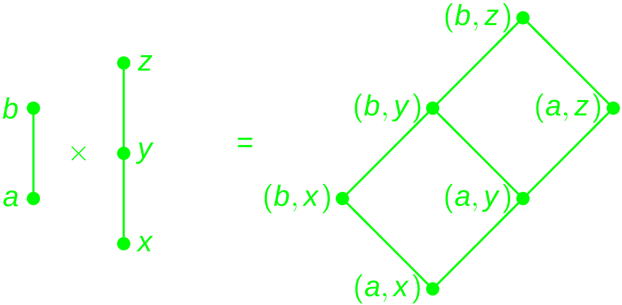
If P and Q are posets, then their *product* is

$$P \times Q = \{(a, x) \ : a \in P, \ x \in Q\}$$

partially ordered by

$$(a, x) \leq_{P \times Q} (b, y) \quad \Longleftrightarrow \quad a \leq_P b \quad \text{and} \quad x \leq_Q y.$$

Example.



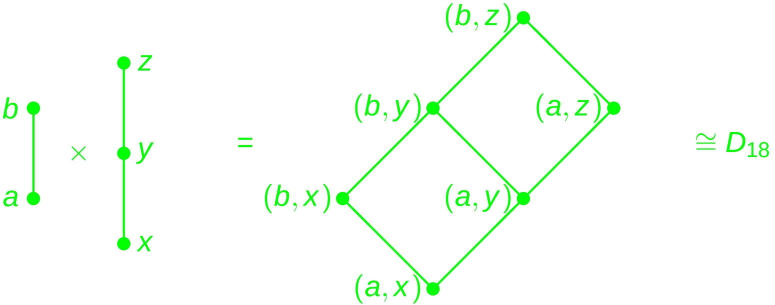
If P and Q are posets, then their *product* is

$$P \times Q = \{(a, x) \ : a \in P, \ x \in Q\}$$

partially ordered by

$$(a, x) \leq_{P \times Q} (b, y) \quad \Longleftrightarrow \quad a \leq_P b \quad \text{and} \quad x \leq_Q y.$$

Example.



If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n *is* $n = p_1^{m_1} \cdots p_k^{m_k}$, *then for the divisor lattice:* $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \leq i \leq n$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \leq i \leq n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \dots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \leq i \leq n$. To show f is order preserving suppose $f(S) = (b_1, \dots, b_n)$ and $f(T) = (c_1, \dots, c_n)$. Now $S \leq T$ in B_n means $S \subseteq T$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \le i \le n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$. Now $S \le T$ in B_n means $S \subseteq T$. Equivalently, $i \in S$ implies $i \in T$ for every $1 \le i \le n$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \leq i \leq n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$. Now $S \leq T$ in B_n means $S \subseteq T$. Equivalently, $i \in S$ implies $i \in T$ for every $1 \leq i \leq n$. So for each $1 \leq i \leq n$ we have $b_i \leq c_i$ in C_1.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \le i \le n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$. Now $S \le T$ in B_n means $S \subseteq T$. Equivalently, $i \in S$ implies $i \in T$ for every $1 \le i \le n$. So for each $1 \le i \le n$ we have $b_i \le c_i$ in C_1. But then $(b_1, \ldots, b_n) \le (c_1, \ldots, c_n)$ in $(C_1)^n$, i.e. $f(S) \le f(T)$.

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \le i \le n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$. Now $S \le T$ in B_n means $S \subseteq T$. Equivalently, $i \in S$ implies $i \in T$ for every $1 \le i \le n$. So for each $1 \le i \le n$ we have $b_i \le c_i$ in C_1. But then $(b_1, \ldots, b_n) \le (c_1, \ldots, c_n)$ in $(C_1)^n$, i.e. $f(S) \le f(T)$. Constructing f^{-1} is done in the obvious way. The proof that f^{-1} is order preserving is just the proof for f read backwards. □

If P is a poset then let $P^n = \overbrace{P \times \cdots \times P}^{n}$.

Proposition

For the Boolean algebra: $B_n \cong (C_1)^n$.
If the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$, then for the divisor lattice: $D_n \cong C_{m_1} \times \cdots \times C_{m_k}$.

Proof for B_n. Since $C_1 = \{0, 1\}$, we define a map $f : B_n \to (C_1)^n$ by

$$f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

for $1 \leq i \leq n$. To show f is order preserving suppose $f(S) = (b_1, \ldots, b_n)$ and $f(T) = (c_1, \ldots, c_n)$. Now $S \leq T$ in B_n means $S \subseteq T$. Equivalently, $i \in S$ implies $i \in T$ for every $1 \leq i \leq n$. So for each $1 \leq i \leq n$ we have $b_i \leq c_i$ in C_1. But then $(b_1, \ldots, b_n) \leq (c_1, \ldots, c_n)$ in $(C_1)^n$, i.e. $f(S) \leq f(T)$. Constructing f^{-1} is done in the obvious way. The proof that f^{-1} is order preserving is just the proof for f read backwards. □

Exercise. Prove the statement for D_n.