Möbius Functions of Embedding Orders

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1. Möbius functions

Let $(P, \leq)$ be a finite poset (partially ordered set). Let $\operatorname{Int} P$ be the set of closed intervals in P:

$$[x, z] = \{y \in P \mid x \leq y \leq z\}.$$

The *incidence algebra* of P is the set

$$I(P) = \{\phi \mid \phi : \operatorname{Int} P \to \mathbb{C}\}$$

under the operations

$$\begin{aligned} (\phi + \psi)(x, z) &= \phi(x, z) + \psi(x, z), \\ (c\phi)(x, z) &= c\phi(x, z), \quad c \in \mathbb{C}, \\ (\phi * \psi)(x, z) &= \sum_{x \leq y \leq z} \phi(x, y)\psi(y, z). \end{aligned}$$

Then $I(P)$ is an algebra with unit the Kronecker delta $\delta(x, z)$ since $\delta * \phi = \phi * \delta = \phi$, e.g.,

$$(\delta * \phi)(x, z) = \sum_{x \leq y \leq z} \delta(x, y)\phi(y, z) = \phi(x, z).$$

Element $\phi \in I(P)$ has convolution inverse ϕ^{-1} iff $\phi(x, x) \neq 0$ for all $x \in P$. The *zeta function* of P is $\zeta(x, z) = 1$ for all $x, z \in P$. The *Möbius function* of P is $\mu = \zeta^{-1}$ so $\zeta * \mu = \delta$ or $\sum_{x \leq y \leq z} \mu(y, z) = \delta(x, z)$ or

$$\mu(x, z) = \begin{cases} 1 & \text{if } x = z, \\ -\displaystyle\sum_{x < y \leq z} \mu(y, z) & \text{if } x < z. \end{cases}$$

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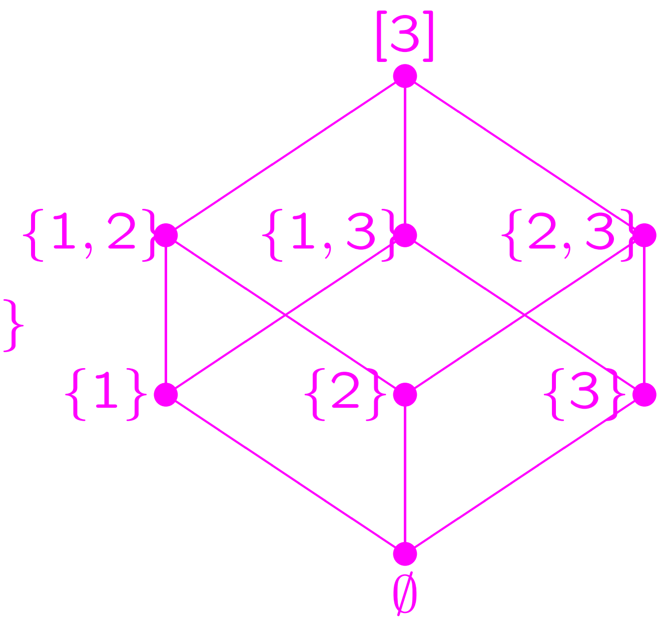
Ex. Let B_n be the Boolean algebra of all subsets of $[n] = \{1, \ldots, n\}$ ordered by inclusion. We compute $\mu(x, [3])$ in B_3, putting the value to the right of x in the following Hasse diagram.



Theorem 1 (Möbius Inversion Thm) *Given any two functions $f, g : P \to \mathbb{C}$, then*

$$f(z) = \sum_{x \leq z} g(x) \quad \forall z \in P$$
$$\iff g(z) = \sum_{x \leq z} \mu(x,z) f(x) \quad \forall z \in P.$$

This Theorem has as corollaries the Principle of Inclusion-Exclusion (for $P = B_n$), the Fundamental Theorem of the Difference Calculus (for P a chain), and the Möbius Inversion Theorem of Number Theory (for P a divisor lattice).

2. Subword order

Let A be an alphabet with $0 \notin A$. Partially order

$$A^* = \{w \mid w \text{ a finite word over } A\}$$

by $v \leq w$ iff v is a subword of w.

Ex. If $w = a\ a\ b\ b\ b\ a\ b\ a$ then $v = a\ b\ b\ a$ is a subword as is shown by the green letters in $w = a\ a\ b\ b\ b\ a\ b\ a$.

Word $\epsilon = \epsilon(1) \ldots \epsilon(n) \in (A \cup 0)^*$ has *support*

$$\text{Supp}\, \epsilon = \{i \mid \epsilon(i) \neq 0\}.$$

An *expansion* of $v \in A^*$ is $\epsilon_v \in (A \cup 0)^*$ such that if one restricts ϵ_v to its support one obtains v. An *embedding* of v into $w = w(1) \ldots w(n)$ is an expansion $\epsilon_v = \epsilon_v(1) \ldots \epsilon_v(n)$ of v such that

$$\epsilon_v(i) = w(i) \qquad \text{for all } i \in \text{Supp}\, \epsilon_v.$$

Note that $v \leq w$ in A^* iff there is an embedding ϵ_v of v into w.

Ex. In the previous example, the expansion of v corresponding to the given subword of w is just $\epsilon_v = a\ 0\ b\ 0\ 0\ 0\ b\ a$.

Given a word $w = w(1) \ldots w(n)$ then a *run of a's* in w is a maximal interval of indices $[r, s]$ such that

$$w(r) = w(r+1) = \cdots = w(s) = a.$$

Ex. $w = a\ a\ b\ b\ b\ a\ b\ a$ has runs of a's: $[1, 2]$, $[6, 6]$, $[8, 8]$; and runs of b's: $[3, 5]$ and $[7, 7]$.

An embedding ϵ_v of v into w is *normal* if for every $a \in A$ and every run $[r, s]$ of a's we have

$$(r, s] \subseteq \operatorname{Supp} \epsilon_v.$$

Ex. In $w = a\ a\ b\ b\ b\ a\ b\ a$ any normal embedding must contain the elements in blue. So there are two normal embeddings of $v = a\ b\ b\ a$, namely $\epsilon_v = 0\ a\ 0\ b\ b\ a\ 0\ 0$ and $\epsilon_v = 0\ a\ 0\ b\ b\ 0\ 0\ a$.

Theorem 2 (Björner) *In A^* we have*

$$\mu(v, w) = (-1)^{|w|-|v|} \binom{w}{v}_n$$

where $|w|$ is the length of w and $\binom{w}{v}_n$ is the number of normal embeddings of v in w.

Ex. We have

$$\mu(abba, aabbbaba) = (-1)^{8-4} \cdot 2 = 2.$$

3. Layered permutations

Let $\mathbb{P}$ denote the positive integers. Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. Then $\pi \in \mathfrak{S}_n$ is *layered* if π has the form

$$\pi = a\ (a-1)\ \dots\ 1\ b\ (b-1)\ \dots (a+1)\ \dots$$

Let $\mathfrak{L}$ be the set of layered permutations partially ordered by pattern containment. Then there is a bijection $\mathfrak{L} \leftrightarrow \mathbb{P}^*$ given by $\pi \leftrightarrow p = p(1) \dots p(k)$ where the $p(i)$ are the layer lengths of π. Under this bijection, the partial order becomes $p \le q$ iff there is an expansion ϵ_p of p which has length $|q|$ and satisfies

$$\epsilon_p(i) \le q(i) \qquad \text{for all } 1 \le i \le |q|.$$

Call such an expansion an *embedding* of p in q.

Ex. If $\pi = 3\ 2\ 1\ 5\ 4$ and $\sigma = 4\ 3\ 2\ 1\ 6\ 5\ 8\ 7$ then one occurrence of π in σ is given by the green numbers in $\sigma = 4\ 3\ 2\ 1\ 6\ 5\ 8\ 7$. In $\mathbb{P}^*$ we have π and σ corresponding to $p = 3\ 2$ and $q = 4\ 2\ 2$, respectively. And the occurrence of p in q corresponds to $\epsilon_p = 3\ 0\ 2$.

An embedding ϵ_p of p in $q \in \mathfrak{S}_n$ is *normal* if

1. For all i, $1 \le i \le n$, we have
$$\epsilon_p(i) = q(i),\ q(i) - 1, \text{ or } 0.$$

2. For every $k \in \mathbb{P}$ and every run $[r, s]$ of k's
   (a) $(r, s] \subseteq \operatorname{Supp} \epsilon_p$ if $k = 1$,
   (b) $r \in \operatorname{Supp} \epsilon_p$ if $k > 1$.

Ex. In $q = 2\ 2\ 1\ 1\ 1\ 3\ 3$ then any normal embedding must support the elements in blue. So there are two normal embeddings of $p = 2\ 1\ 1\ 1\ 3$, namely $\epsilon_p = 2\ 1\ 0\ 1\ 1\ 3\ 0$ and $\epsilon_p = 2\ 0\ 1\ 1\ 1\ 3\ 0$.

The *sign* of a normal embedding ϵ_p of p in q is

$$(-1)^{\#\text{ of } i \text{ where } \epsilon_p(i) = q(i) - 1}.$$

The exponent is the *defect* $d(\epsilon_p)$.

Theorem 3 (S-V) *In $\mathfrak{L}$ we have*

$$\mu(p, q) = \sum_{\epsilon_p} (-1)^{d(\epsilon_p)}$$

summed over all normal embeddings ϵ_p of p in q.

Ex. We have

$$\mu(21113, 2211133) = (-1)^2 + (-1)^0 = 2.$$

4. Further work

A. Topology of $\mathfrak{L}$. If P is a poset then $[x,z] \subseteq P$ has *order complex*

$$\Delta(x,z) = \{c \mid c \text{ a chain in } (x,z)\}.$$

So $\Delta(x,z)$ is a simplicial complex with *reduced Euler characteristic*

$$\tilde{\chi}(\Delta(x,z)) := \sum_{i \geq -1} (-1)^i \operatorname{rk} \tilde{H}_i(\Delta(x,z)) = \mu(x,z).$$

Theorem 4 (Björner) *In A^*, the interval $[v,w]$ is lexicographically shellable for all v, w. And*

$$\operatorname{rk} \tilde{H}_i(\Delta(v,w)) = \begin{cases} \binom{w}{v}_n & \text{if } i = |w| - |v| - 2, \\ 0 & \text{else.} \end{cases}$$

In $\mathfrak{L}$, $[p,q]$ is *not* always shellable. But Forman developed a discrete analogue of Morse Theory to compute the homology of any CW-complex by collapsing it onto a subcomplex of critical cells. Babson & Hersh showed how any lexicographic ordering of the maximal chains of an interval gives rise to the critical cells of a Morse function.

Conjecture 5 *In $\mathcal{L}$ there is a Morse function for $[p,q]$ with a single critical cell of dimension $d(\epsilon_p)$ for each normal embedding ϵ_p of p in q.*

B. Embedding orders. Let P be any poset. Take $0 \notin P$ and set $0 < x$ for all $x \in P$. Partially order P^* by $p \le q$ in P^* iff there is an expansion ϵ_p of length $|q|$ with

$$\epsilon_p(i) \le q(i) \qquad \text{for all } 1 \le i \le |q|.$$

Call this the *embedding order* on P^*.

Call P a *rooted forest* if each component of the Hasse diagram of P is a tree with a unique minimal element. Then there is a notion of normal embedding in P^* where minimal elements play the role of $q(i) = 1$, nonminimal elements play the role of $q(i) > 1$, and the element adjacent to $q(i)$ on then unique $q(i)$-root path plays the role of $q(i) - 1$.

Conjecture 6 *Let P be a rooted forest. Then in P^* we have*

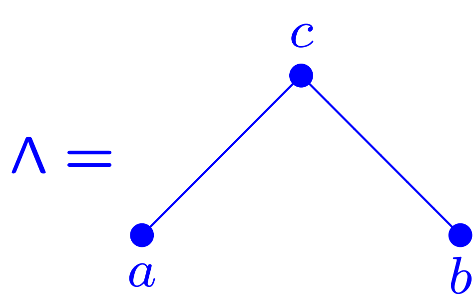
$$\mu(p,q) = \sum_{\epsilon_p} (-1)^{d(\epsilon_p)}$$

summed over all normal embeddings ϵ_p of p in q.

Note that if this conjecture is true then the theorems for A^* or $\mathfrak{L}$ are the special cases where P is an antichain or a chain, respectively.

C. Other orders. Let $\mathfrak{S}$ be the set of all permutations ordered by pattern containment. What is $\mu(p, q)$ for $p, q \in \mathfrak{S}$?

What about P^* for any poset P (not just rooted forests)? The simplest such poset is



Let a^j denote the word in Λ^* consisting of j copies of a and similarly for the other elements of Λ. Let $T_n(x)$ denote the *nth Chebyshev polynomial of the first kind*.

Conjecture 7 *If $j, k \geq 0$ then $\mu(a^j, c^k)$ is the coefficient of x^{k-j} in $T_{k+j}(x)$.*