The k-consecutive Arrangements

Bruce E. Sagan
(joint with William Brockman)
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

1. The coordinate case
   (a) Möbius functions & NBB bases
   (b) Characteristic polynomials & lattice points
2. The type A case
   (a) Möbius functions again
   (b) Characteristic polynomials again
3. Comments & open problems

1. The coordinate case

For $i, j, n \in \mathbb{Z}_{\geq 0}$ we will use the notation

$$[n] = \{1, 2, \ldots, n\} \quad \text{and} \quad [i, j] = \{i, i+1, \ldots, j\}.$$

Define the *k-consecutive coordinate arrangement*, $\mathcal{K}_{n:k}$, as all subspaces of $\mathbb{R}^n$ of the form

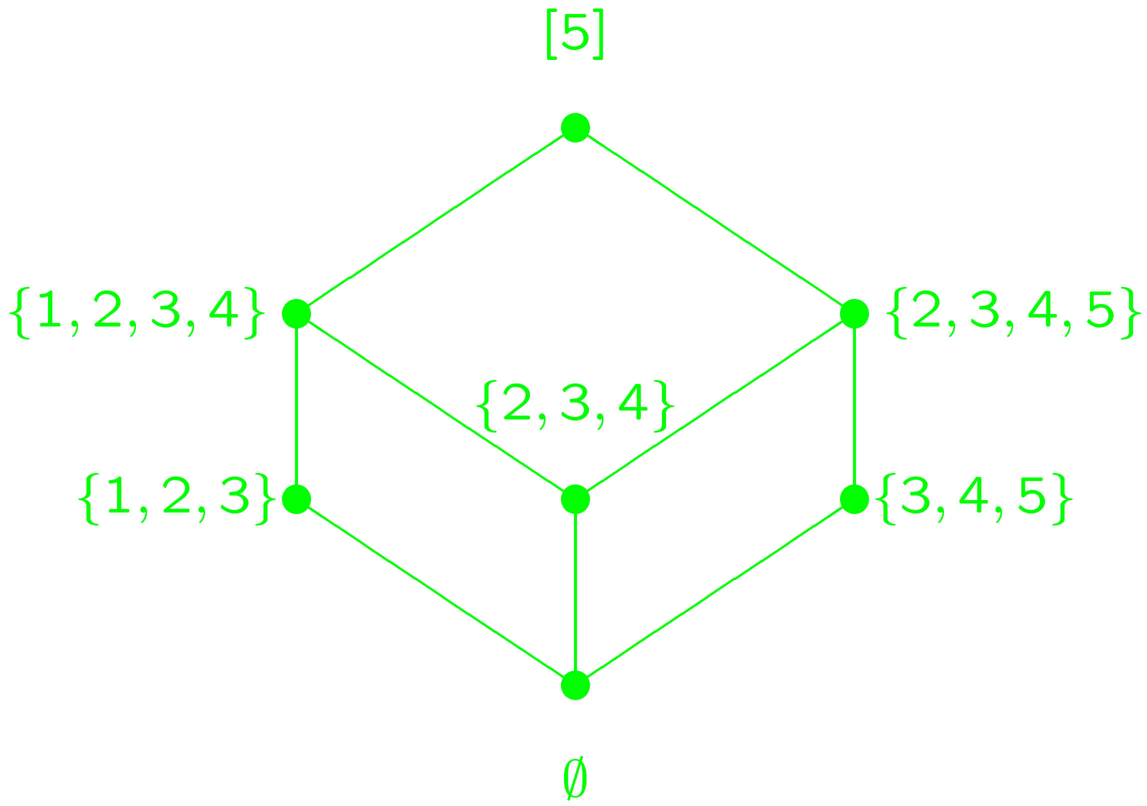
$$x_i = x_{i+1} = \ldots = x_{i+k-1} = 0, \qquad 1 \leq i \leq n-k+1.$$

The intersection lattice $L(\mathcal{K}_{n:k})$ is isomorphic to the poset $B_{n:k}$ generated by taking joins of intervals

$$[i, i+k-1], \qquad 1 \leq i \leq n-k+1$$

in the Boolean algebra of subsets of $[n]$. As an example

$B_{5:3} =$



1 (a). Möbius functions & NBB bases

Let $(L, \leq)$ be a finite lattice with minimum $\hat{0}$, maximum $\hat{1}$, and join (least upper bound) operation $\vee$. Let $\mu : L \to \mathbf{Z}$ be L's Möbius function which is the unique function satisfying

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0}x}.$$

Let $A(L)$ be the atom set of L and put an arbitrary partial order $\trianglelefteq$ on $A(L)$. Then $D \subseteq A(L)$ is *bounded below (BB)* if, for every $d \in D$ there is an $a \in A(L)$ such that

$$\begin{array}{lcll} a & \lhd & d & \quad\text{and} \\ a & < & \bigvee D. & \end{array}$$

Then $B \subseteq A(L)$ is an *NBB base of x* if $x = \bigvee B$ and B does not contain any D which is BB.

Theorem 1 (Blass-S) *Let L be any finite lattice and let $\trianglelefteq$ be any partial order on $A(L)$. Then for all $x \in L$*

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x. ■

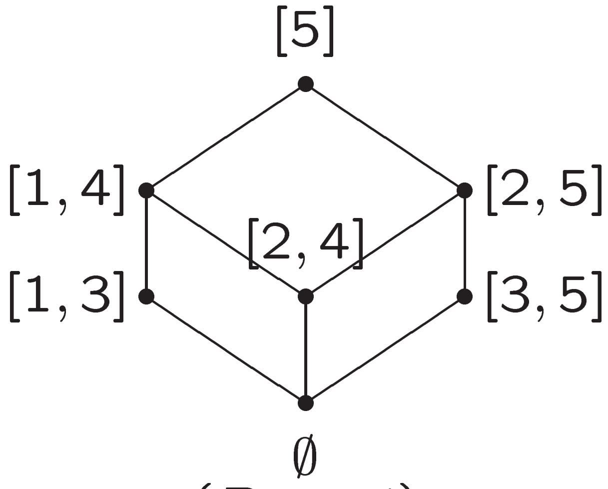
$D \subseteq A(L)$ is *BB* if for all d $\in$ D there is $a \in A(L)$ s.t.

$$a \lhd d \qquad \text{and} \qquad a < \bigvee D.$$

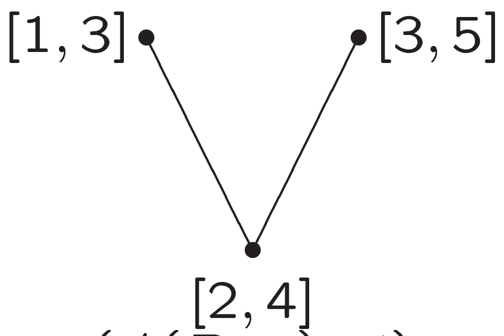
Theorem 1 (Blass-S) *For all $x \in L$*

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x. ■



$(B_{n:k}, \leq)$



$(A(B_{n:k}), \unlhd)$

Note that from the definition of BB

1. No set containing a min. element of $\unlhd$ is BB.
2. No set with at most one element is BB.

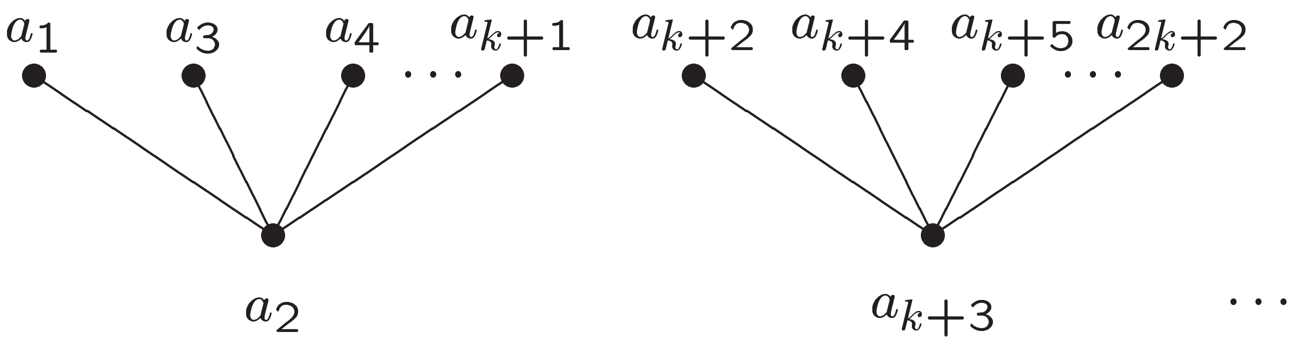
So for the given $\unlhd$ in $A(B_{n:k})$, the only possible BB set is $\{[1,3],[3,5]\}$. It is since $[2,4] \lhd [1,3],[3,5]$ and $[2,4] < [1,3] \vee [3,5] = [5]$.

x	$\emptyset$	$[1,3]$	$[1,4]$	$[1,5]$
NBB bases	$\emptyset$	$\{[1,3]\}$	$\{[1,3],[2,4]\}$	none
$\mu(x)$	$(-1)^0$	$(-1)^1$	$(-1)^2$	0

Corollary 2 (Greene) *In $B_{n:k}$ we have*

$$\mu([n]) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{k+1}, \\ -1 & \text{if } n \equiv -1 \pmod{k+1}, \\ 0 & \text{else.} \end{cases}$$

Proof. Let the atoms of $B_{n:k}$ be $a_1, \ldots, a_{n-k+1}$ where $a_i = [i, i+k-1]$. Define $\trianglelefteq$ by



Let B be an NBB base of $[n]$ if it exists. Then $a_1 \in B$ since a_1 is the only atom containing 1. So none of $a_3, \ldots, a_{k+1}$ is in B since any of these atoms forms a BB set with a_1. The only available atom remaining which contains $k+1$ is a_2, forcing $a_2 \in B$. Iterating this argument we find that if B exists then it must be unique and

$$B = \{a_1, a_2, a_{k+2}, a_{k+3}, \ldots\}.$$

If $n \equiv 0$ or $-1 \pmod{k+1}$ then $\bigvee B = [n]$ so we have a base of even or odd cardinality, respectively. Otherwise $a_{n-k+1} \notin B$ and since this is the only atom containing n, B is not a base for $[n]$. ■

1 (b) Characteristic polynomials & lattice points

The *characteristic polynomial* of arrangement $\mathcal{A}$ is

$$\chi(\mathcal{A},t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

For example, if $\mathcal{A} = \{x = 0,\ y = 0\} = \mathcal{K}_{2:1}$ then

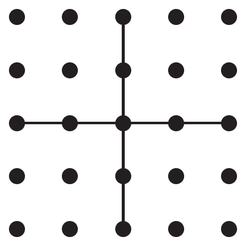
X	$\mathbb{R}^2$	$x = 0$	$y = 0$	$x = y = 0$
$\mu(x)$	1	-1	-1	1
$t^{\dim x}$	t^2	t	t	1

so $\chi(\mathcal{K}_{2:1}, t) = t^2 - t - t + 1 = (t-1)^2$.

Theorem 3 (Crapo-Rota, Terao, Athanasiadis) *If $\mathcal{A}$ is a subspace arrangement in $\mathbb{R}^n$ defined over $\mathbb{Z}$ and hence over $\mathbb{F}_p$, then for large enough primes p*

$$\chi(\mathcal{A}, p) = |\mathbb{F}_p^n \setminus \bigcup \mathcal{A}|. \quad \blacksquare$$

$\mathcal{A} = \mathcal{K}_{2:1}$
$p = 5$
$\mathbb{Z}_5 = \{-2, -1, 0, 1, 2\}$



Then removing the lines of $\mathcal{K}_{2:1}$ from the plane $\mathbb{F}_5^2$ leaves $|\mathbb{F}_5^2 \setminus \bigcup \mathcal{K}_{2:1}| = 16$ lattice points. Also

$$\chi(\mathcal{K}_{2:1}, 5) = (5-1)^2 = 16.$$

Theorem 4 (Crapo-Rota, Terao, Athanasiadis) *If $\mathcal{A}$ is a subspace arrangement in $\mathbb{R}^n$ defined over $\mathbb{Z}$ and hence over $\mathbb{F}_p$, then for large enough primes p*

$$\chi(\mathcal{A}, p) = |\mathbb{F}_p^n \setminus \bigcup \mathcal{A}|. \quad \blacksquare$$

$$\binom{n}{i}_k := \#\text{ of } S \subseteq [n],\ |S| = i,\text{ no } k \text{ consecutive.}$$

For example, $\binom{6}{3}_2 = 4$ counting

$$\{1,3,5\};\ \{1,3,6\};\ \{1,4,6\};\ \{2,4,6\}.$$

Proposition 5 *We have*

$$\chi(\mathcal{K}_{n:k}, t) = \sum_i \binom{n}{i}_k (t-1)^{n-i}, \tag{1}$$

$$(t-1)^{\lfloor n/k \rfloor} \mid \chi(\mathcal{K}_{n:t}, t). \tag{2}$$

Proof. Let $p >> 0$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_p^n \setminus \bigcup \mathcal{K}_{n:k}$. Then if i of the coordinates are to be zero, there are $\binom{n}{i}_k$ ways to pick these x_j. Then the remaining $n-i$ nonzero coordinates can be chosen in a total of $(p-1)^{n-i}$ ways. Summing on i, (1) follows.

A largest subset of $[n]$ with no k consecutive is

$$[n] \setminus \{k, 2k, 3k, \ldots\}.$$

So $\binom{n}{i}_k = 0$ if $n - i < \lfloor n/k \rfloor$. This gives (2). $\blacksquare$

The type A case

Define the *k-consecutive type A arrangement*, $\mathcal{A}_{n:k}$, as all subspaces of $\mathbb{R}^n$ of the form

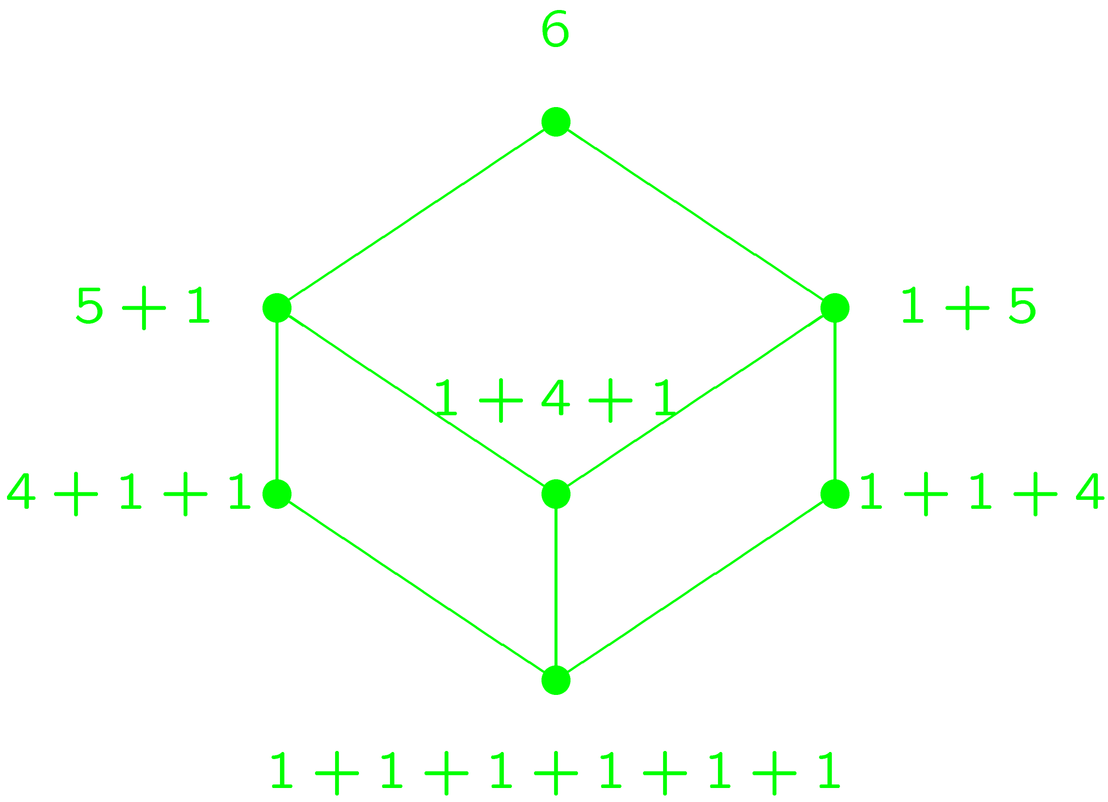
$$x_i = x_{i+1} = \ldots = x_{i+k-1}, \qquad 1 \leq i \leq n-k+1.$$

The intersection lattice $L(\mathcal{A}_{n:k})$ is isomorphic to the poset $C_{n:k}$ generated by taking joins of compositions

$$k+1+\cdots+1, 1+k+1+\cdots+1, \ldots, 1+\cdots+1+k.$$

in the posets of all compositions of n ordered by refinement. As an example

$C_{6:4} =$



Define a lattice

$$B^*_{n:k} = \{[n] \setminus S \ \ : \ \ S \in B_{n:k}\}$$

ordered by *reverse* inclusion. Define two functions $\alpha : B_{n:k} \to B^*_{n:k}$ and $\beta : B^*_{n:k} \to C_{n+1:k+1}$ by

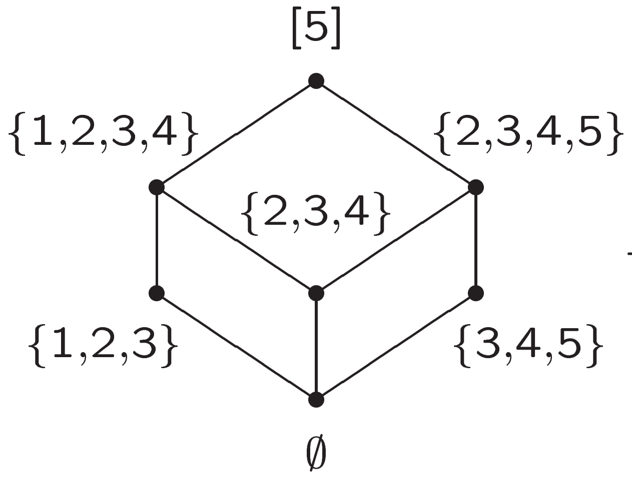
$$S \xrightarrow{\alpha} [n] \setminus S,$$

$$\{n_1, \ldots, n_l\}_< \xrightarrow{\beta} n_1 + (n_2 - n_1) + \cdots + (n + 1 - n_l).$$

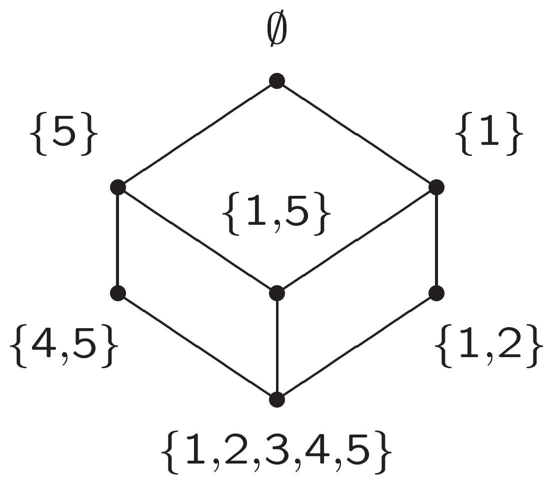
Then α, β are lattice isomorphisms so

$$C_{n+1:k+1} \cong B_{n:k}.$$

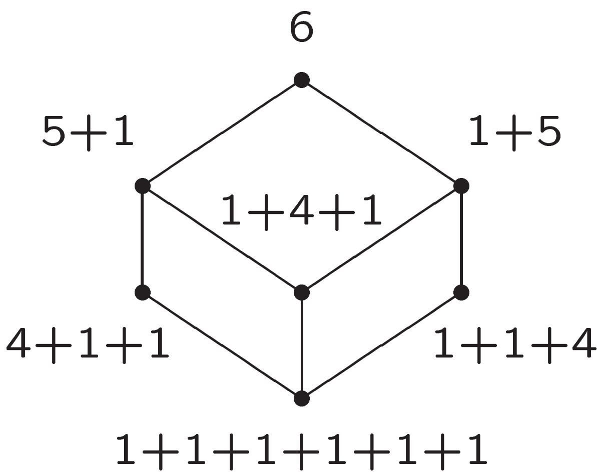
For example



$\xrightarrow{\alpha}$



$\xrightarrow{\beta}$



2 (a) Möbius functions again

Theorem 6 *Let $f(x) = \sum_{n\geq 1} a_n x^n$, then we have*

$$\frac{1}{1-f(x)} = \sum_{n\geq 0}\left(\sum_{n_1+n_2+\cdots+n_l=n} a_{n_1}a_{n_2}\cdots a_{n_l}\right)x^n. \qquad \blacksquare$$

For the $C_{n:k}$ with k fixed, define $m_k(x) = \sum_{n\geq 1} \mu(n)x^n$.

Corollary 7 *We have $m_k(x) = \dfrac{x-x^k}{1-x^k}$.*
Equivalently, in $C_{n:k}$ we have

$$\mu([n]) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{k}, \\ -1 & \text{if } n \equiv 0 \pmod{k}, \\ 0 & \text{else.} \end{cases}$$

Proof. We let $\eta = n_1 + \cdots + n_l \in C_{n:k}$ and define $C_\eta := \{\lambda \in C_{n:k} \ : \ \lambda \leq \eta\} \cong C_{n_1:k} \times \cdots \times C_{n_l:k}$. So $\mu(\eta) = \mu(n_1)\cdots\mu(n_l)$ and

$$\begin{aligned}\frac{1}{1-m_k(x)} &= \sum_{n\geq 0}\left(\sum_{\eta\in C_{n:k}} \mu(\eta)\right)x^n \text{ (Theorem 6)} \\ &= 1 + x + x^2 + \cdots + x^{k-1}\end{aligned}$$

since the inner sum is 1 or 0 depending on whether $\hat{0} = \hat{1}$ or not in $C_{n:k}$. Solving for $m_k(x)$, simplifying, & taking the coefficient of x^n finishes the proof. $\blacksquare$

2 (b) Characteristic polynomials again

If $S \in B_{n-1:k-1}$ then $\beta\alpha(S) \in C_{n:k}$ and

$$\dim \beta\alpha(S) = 1 + \dim S.$$

Corollary 8 *We have*

$$\begin{aligned} \chi(\mathcal{A}_{n:k}, t) &= t\chi(\mathcal{K}_{n-1:k-1}, t) \\ &= \sum_i \binom{n-1}{i}_{k-1} t(t-1)^{n-i-1}. \quad \blacksquare \end{aligned}$$

$$\begin{aligned} \langle t \rangle_i &:= t(t-1)(t-2)\cdots(t-i+1), \\ S_k(n:i) &:= \#\text{ of partitions } B_1/\ldots/B_i \text{ of } [n]\text{, no} \\ & \qquad B_j \text{ containing } k \text{ consecutive integers.} \end{aligned}$$

For example, $S_3(4:2) = 5$ counting

$$1,2/3,4;\ \ 1,3/2,4;\ \ 1,4/2,3;\ \ 1,2,4/3;\ \ 1,3,4/2.$$

Proposition 9 *We have*

$$\chi(\mathcal{A}_{n:k}, t) = \sum_i S_k(n:i)\langle t \rangle_i.$$

Proof. Let $p >> 0$ and $\mathbf{x} \in \mathbb{F}_p^n \setminus \bigcup \mathcal{A}_{n:k}$. If $\mathbf{x}$ contains i different coordinates, then there are $\langle p \rangle_i$ ways to pick the values to be used and $S_k(n:i)$ ways to distribute these values among the coordinates. $\blacksquare$

Note that $S_k(n:i) > 0$ for $i > 1$ so no nice divisibility relation can be derived.

3. Comments & open problems

a. Coefficients. $\binom{n}{i}_k$ and $S_k(n:i)$ have interesting properties. For example we have

$$\binom{n}{i}_k = \binom{n}{i} \text{ and } S_k(n:i) = S(n,k) \text{ for } 0 \le n < k.$$

For the $\binom{n}{i}_k$ and small k

$$\begin{aligned}
\binom{n}{i}_1 &= \delta_{i,0} \quad \text{(Kronecker)},\\
\binom{n}{i}_2 &= \binom{n-i+1}{i},\\
\binom{n}{i}_3 &= \sum_m \binom{m}{i-m-2}\binom{n-i+2}{m+2} + \binom{m}{i-m-3}\binom{n-i+1}{m+2} \ (i \ge 3).
\end{aligned}$$

We also have the recursion

$$\binom{n}{i}_k + \binom{n-k-1}{i-k}_k = \binom{n-1}{i}_k + \binom{n-1}{i-1}_k.$$

To prove this, consider

$$\binom{[n]}{i}_k := \{S \subseteq [n] : |S| = i, \text{ no } k \text{ consecutive}\} = \mathcal{S}_1 \uplus \mathcal{S}_2$$

where $\mathcal{S}_1 = \{S \in \binom{[n]}{i}_k \ : \ n \notin S\}$. Also

$$\binom{[n-1]}{i-1}_k = \mathcal{T}_1 \uplus \mathcal{T}_2$$

where $\mathcal{T}_1 = \{T \in \binom{[n-1]}{i-1}_k \ : \ \{n-k+1,\ldots,n\} \not\subseteq T\}$.

Then there are bijections

$$\mathcal{S}_1 \leftrightarrow \binom{[n-1]}{i}_k,\ \mathcal{S}_2 \leftrightarrow \mathcal{T}_1,\ \binom{[n-k-1]}{i-k}_k \leftrightarrow \mathcal{T}_2. \quad \blacksquare$$

b. Topology. Let S^d be the sphere of dimension d and let $\Delta(L)$ be the order complex of lattice L. Using non-pure lexicographic shellings Björner and Wachs proved

Theorem 10 *We have*

$$\Delta(C_{n:k}) \simeq \begin{cases} S^{2(n-1)/k-2} & \textit{if } n \equiv 1 \pmod{k}, \\ S^{2n/k-3} & \textit{if } n \equiv 0 \pmod{k}, \\ \textit{point} & \textit{else.} \end{cases} \quad \blacksquare$$

A subset of an NBB set is NBB, so let NBB(L) be the simplicial complex of NBB bases of all of $x \in L, x \neq \hat{1}$. Segev has shown

Theorem 11 *We have*

$$\Delta(L) \simeq \mathrm{NBB}(L). \quad \blacksquare$$

It would be interesting to derive Theorem 10 from Theorem 11. This is non-trivial since the NBB sets of L do not form a matroid.

One can obtain a basis for the homology of a geometric lattice L as follows. Given an NBC base $B = \{a_1, a_2, \ldots, a_n\}$ for $\hat{1}$ let

$$\rho_B = \sum_{\pi \in \mathfrak{S}_n} (-1)^\pi [a_{\pi 1},\ a_{\pi 1 \vee \pi 2},\ \ldots,\ a_{\pi 1 \vee \cdots \vee \pi(n-1)}].$$

For example, let $L = B_3$ (Boolean algebra) and also let $B = \{\{1\},\ \{2\},\ \{3\}\}$. Then

$$\begin{array}{rcl} \pi & : & \epsilon \qquad (1,2) \quad (1,3) \quad (2,3) \quad (1,2,3) \quad (1,3,2) \\ \rho_B & = & [1,12]-[2,12]-[3,23]-[1,13]+[2,23]+[3,13] \\ & = & [1,12]+[12,2]+[2,23]+[23,3]+[3,13]+[13,1] \end{array}$$

so $\partial \rho_B = 0$.

Theorem 12 (Björner) *If L is a geometric lattice of rank n then*

$$\{\rho_B \mid B \text{ is an NBC base of } \hat{1}\}$$

is a basis for $H_{n-2}(\Delta L, \mathbb{Z})$, the only non-zero homology group. ■

Is there an NBB analog of this result?

c. Other types. We can define k-consecutive analogs for other Coxeter arrangements as has been done by Björner and Sagan in the k-equal case. For example, the type B arrangement is $\mathcal{B}_{n:k}$ with subspaces

$$\{\epsilon_i x_i = \ldots = \epsilon_{i+k-1} x_{i+k-1} \mid 1 \leq i \leq n-k+1\}$$
$$\cup \{x_j = 0 \mid 1 \leq j \leq n\}.$$

Theorem 13 *We have*

$$\sum_{n \geq 0} \chi(L(\mathcal{B}_{n:k}), q) x^n$$
$$= 1 + \frac{(q-1)x(1-2^{k-1}x^{k-1})}{1-2x-(q-3)x(1-2^{k-1}x^{k-1})}.$$

In particular for $k = 2$

$$\sum_{n \geq 0} \chi(L(\mathcal{B}_{n:2}), q) x^n = 1 + \frac{(q-1)x}{1-(q-3)x}. \qquad \blacksquare$$

The Möbius function of $L(\mathcal{B}_{n:2})$ can be computed using NBB bases and Zaslavsky's theory of signed graphs.

d. k-circular arrangements. Following a suggestion of Athanasiadis, define the *k-circular coordinate arrangement*, $\mathcal{K}^{\circ}_{n:k}$, as all subspaces of $\mathbb{R}^n$ of the form

$$x_i = x_{i+1} = \ldots = x_{i+k-1} = 0, \qquad 1 \le i \le n$$

where the subscripts are taken modulo n. Then $L(\mathcal{K}^{\circ}_{n:k})$ is isomorphic to the poset $B^{\circ}_{n:k}$ generated by taking joins of intervals

$$[i, i+k-1], \qquad 1 \le i \le n$$

($i+k-1$ taken modulo n) in the Boolean algebra of subsets of $[n]$.

Proposition 14 *In $B^{\circ}_{n:k}$, $n \ge k$, we have*

$$\mu([n]) = \begin{cases} k & \text{if } n \equiv 0 \pmod{k+1}, \\ -1 & \text{else.} \end{cases}$$

and

$$\chi(\mathcal{K}^{\circ}_{n:k}, t) = \sum_i \binom{n}{i}^{\circ}_k (t-1)^{n-i},$$

where $\binom{n}{i}^{\circ}_k$ is the number of $S \subseteq [n]$, $|S| = i$, with no k circularly consecutive. ■