Increasing spanning forests

Joshua Hallam
Wake Forest University

Jeremy Martin
University of Kansas

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

April 9, 2016

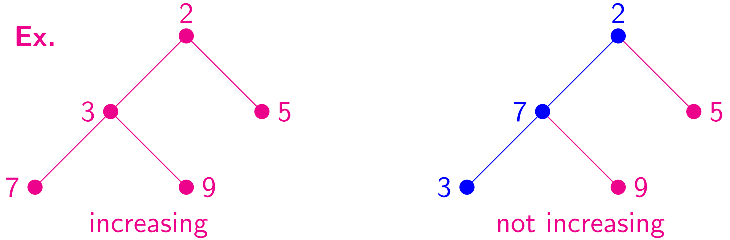
The factorization theorem

Connection with the chromatic polynomial

Comments and future work

All graphs $G = (V, E)$ will have V a set of positive integers. A tree T is *increasing* if the vertices along any path starting at the minimum vertex form an increasing sequence.

Ex.



A forest is *increasing* if each of its component trees is increasing. For any graph G, let

$$\mathrm{isf}_m(G) = \#\text{ of increasing spanning forests of } G \text{ with } m \text{ edges.}$$

Any isolated vertex or edge is an increasing tree, so

$$\mathrm{isf}_0(G) = 1 \quad \text{and} \quad \mathrm{isf}_1(G) = |E|.$$

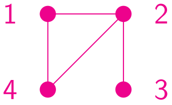
If G has n vertices, then let

$$\mathrm{ISF}(G) = \mathrm{ISF}(G,t) = \sum_{m \geq 0} (-1)^m \,\mathrm{isf}_m(G) t^{n-m}.$$

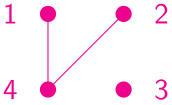
$\mathrm{isf}_m(G) = \#$ of increasing spanning forests of G with m edges.

$$\mathrm{ISF}(G) = \mathrm{ISF}(G,t) = \sum_{m \geq 0} (-1)^m \, \mathrm{isf}_m(G) t^{n-m}.$$

Ex. $G =$



not increasing:

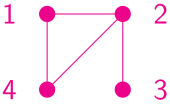


$$\mathrm{isf}_0(G) = 1$$
$$\mathrm{isf}_1(G) = |E| = 4$$
$$\mathrm{isf}_2(G) = \binom{4}{2} - 1 = 5$$
$$\mathrm{isf}_3(G) = \binom{4}{3} - 2 = 2$$
$$\mathrm{isf}_4(G) = 0$$
$$\mathrm{ISF}(G) = t^4 - 4t^3 + 5t^2 - 2t = t(t-1)^2(t-2).$$

Let $[n] = \{1, 2, \ldots, n\}$. All graphs will have vertex set $V = [n]$.
For $j \in [n]$ define

$$E_j = \{ij \in E \;:\; i < j\}.$$

Ex.

$G =$ 

$$\therefore\ E_1 = \emptyset, \quad E_2 = \{12\}, \quad E_3 = \{23\}, \quad E_4 = \{14, 24\},$$

and

$$(t - |E_1|)(t - |E_2|)(t - |E_3|)(t - |E_4|) = t(t-1)^2(t-2) = \mathrm{ISF}(G).$$

Theorem (Hallam-S)

Let G have $V = [n]$ and E_j as defined above. Then

$$\mathrm{ISF}(G; t) = \prod_{j=1}^{n} (t - |E_j|).$$

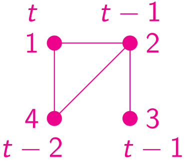
For a positive integer t, a *proper coloring* of $G = (V, E)$ is $c : V \to \{c_1, \ldots, c_t\}$ such that

$$ij \in E \implies c(i) \neq c(j).$$

The *chromatic polynomial* of G is

$$P(G) = P(G; t) = \# \text{ of proper colorings } c : V \to \{c_1, \ldots, c_t\}.$$

Ex. Coloring vertices in the order $1, 2, 3, 4$ gives choices

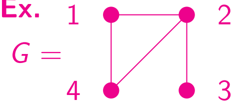


$$P(G; t) = t(t-1)(t-1)(t-2)$$
$$= \mathrm{ISF}(G; t)$$

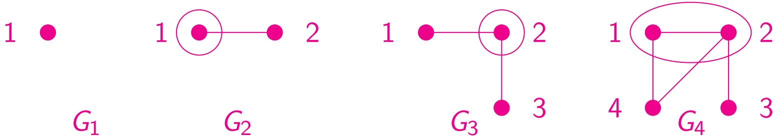
Note 1. $P(G; t)$ is always a polynomial in t.
2. We can not always have $P(G; t) = \mathrm{ISF}(G; t)$ since $P(G; t)$ does not always factor with integral roots.

If G is a graph and $W \subseteq V$, let $G[W]$ denote the induced subgraph of G with vertex set W. Say that an ordering $v_1, \ldots, v_n$ of V is a *perfect elimination ordering (peo)* if, for all j, the neighbors of v_j in $G_j := G[v_1, \ldots, v_j]$ form a clique (complete subgraph).

Ex.



Consider the ordering $1, 2, 3, 4$.
We circle the neighbors of v_j in G_j.



If G has a peo and n_j is the number of neighbors of v_j in G_j then

$$P(G; t) = \prod_{j=1}^{n} (t - n_j).$$

Theorem (Hallam-S)

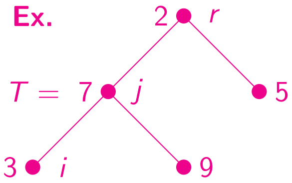
Let G be a graph with $V = [n]$. Then $P(G; t) = \mathrm{ISF}(G; t)$ if and only if $1, \ldots, n$ is a peo of G.

1. Simplicial complexes. A simplicial complex is an object formed by gluing together tetrahedra of various dimensions. A graph is a simplicial complex of dimension 1 since it is formed by gluing together edges. Hallam, Martin, and S have analogues of these results for general simplicial complexes.

2. Inversions. Let T be a tree with minimum vertex r. An *inversion* of T is a pair of vertices $j > i$ such that j is on the unique r–i path. Let

$$\text{inv } T = \# \text{ of inversions of } T.$$

Ex.



$\text{inv } T = 1$

Note that T is increasing if and only if $\text{inv } T = 0$. What can be said about for more inversions? Hallam, Martin, and S have some preliminary results for one inversion.

THANKS FOR LISTENING!