Infinite log-concavity

Peter R. W. McNamara
Department of Mathematics, Bucknell University,
Lewisburg, PA 17837, USA, peter.mcnamara@bucknell.edu

and

Bruce E. Sagan
Department of Mathematics, Michigan State University,
East Lansing, MI 48824-1027, sagan@math.msu.edu
www.math.msu.edu/~sagan

October 17, 2008

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

Outline

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$.

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$.

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Example. $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is log-concave.

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Example. $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is log-concave.

Define the *$\mathcal{L}$-operator* on sequences by

$$\mathcal{L}(a_k) = (a_k^2 - a_{k-1}a_{k+1}).$$

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Example. $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is log-concave.

Define the *$\mathcal{L}$-operator* on sequences by

$$\mathcal{L}(a_k) = (a_k^2 - a_{k-1}a_{k+1}).$$

So (a_k) is log-concave iff $\mathcal{L}(a_k)$ is nonnegative.

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Example. $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is log-concave.

Define the *$\mathcal{L}$-operator* on sequences by

$$\mathcal{L}(a_k) = (a_k^2 - a_{k-1}a_{k+1}).$$

So (a_k) is log-concave iff $\mathcal{L}(a_k)$ is nonnegative. Call (a_k) *infinitely log-concave* if

$$\mathcal{L}^i(a_k) \text{ is nonnegative for all } i \geq 0.$$

Let

$$(a_k) = a_0, a_1, a_2, \ldots$$

be a sequence of nonnegative reals, written $(a_k) \geq 0$. We set $a_k = 0$ for $k < 0$. Call (a_k) *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1} \text{ for all } k.$$

Example. $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is log-concave.

Define the *$\mathcal{L}$-operator* on sequences by

$$\mathcal{L}(a_k) = (a_k^2 - a_{k-1}a_{k+1}).$$

So (a_k) is log-concave iff $\mathcal{L}(a_k)$ is nonnegative. Call (a_k) *infinitely log-concave* if

$$\mathcal{L}^i(a_k) \text{ is nonnegative for all } i \geq 0.$$

Conjecture (Boros-Moll)

Sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is infinitely log-concave for all $n \geq 0$.

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \stackrel{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \qquad (1)$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \stackrel{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \stackrel{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2/r$.

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2/r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \overset{?}{\leq} a_k^4.$$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2/r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \overset{?}{\leq} a_k^4.$$

So suffices $\frac{r-1}{r^2} + \frac{2}{r} = 1$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \qquad (1)$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2 / r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \overset{?}{\leq} a_k^4.$$

So suffices $\frac{r-1}{r^2} + \frac{2}{r} = 1 \iff r^2 - 3r + 1 = 0$

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \quad (1)$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2/r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \overset{?}{\leq} a_k^4.$$

So suffices $\frac{r-1}{r^2} + \frac{2}{r} = 1 \iff r^2 - 3r + 1 = 0 \iff r = \frac{3 \pm \sqrt{5}}{2}$.

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \stackrel{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \stackrel{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2/r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \stackrel{?}{\leq} a_k^4.$$

So suffices $\frac{r-1}{r^2} + \frac{2}{r} = 1 \iff r^2 - 3r + 1 = 0 \iff r = \frac{3 \pm \sqrt{5}}{2}$.

Theorem

(i) If $(a_k) \geq 0$ is r-factor log-concave, $r = \frac{3+\sqrt{5}}{2}$, so is $\mathcal{L}(a_k)$.

Call (a_k) *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \text{ for all } k \geq 0. \tag{1}$$

Want $r > 1$ so that if (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$:

$$(a_k^2 - a_{k-1}a_{k+1})^2 \overset{?}{\geq} r(a_{k-1}^2 - a_{k-2}a_k)(a_{k+1}^2 - a_k a_{k+2}).$$

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_{k+1}a_k^2 \overset{?}{\leq} a_k^4 + \underbrace{r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}}_{\geq 0}.$$

From (1): $a_{k-1}a_{k+1} \leq a_k^2 / r$. Plug in above and drop ≥ 0 terms:

$$\frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4 \overset{?}{\leq} a_k^4.$$

So suffices $\frac{r-1}{r^2} + \frac{2}{r} = 1 \iff r^2 - 3r + 1 = 0 \iff r = \frac{3 \pm \sqrt{5}}{2}$.

Theorem

(i) If $(a_k) \geq 0$ is r-factor log-concave, $r = \frac{3+\sqrt{5}}{2}$, so is $\mathcal{L}(a_k)$.

(ii) $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is infinitely log-concave for $n \leq 1450$. □

Outline

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.

Example. $\mathcal{L}_k \binom{n}{k}$ is for rows, $\mathcal{L}_n \binom{n}{k}$ is for columns.

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.
Example. $\mathcal{L}_k \binom{n}{k}$ is for rows, $\mathcal{L}_n \binom{n}{k}$ is for columns.
Also, let $L(a_k)$ be the kth term of $\mathcal{L}(a_k)$ and similarly with subscripts.

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.

Example. $\mathcal{L}_k\binom{n}{k}$ is for rows, $\mathcal{L}_n\binom{n}{k}$ is for columns.

Also, let $L(a_k)$ be the kth term of $\mathcal{L}(a_k)$ and similarly with subscripts.

Conjecture

The sequence $\binom{n}{k}_{n\geq 0}$ *is infinitely log-concave for all* $k \geq 0$*.*

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.

Example. $\mathcal{L}_k \binom{n}{k}$ is for rows, $\mathcal{L}_n \binom{n}{k}$ is for columns.

Also, let $L(a_k)$ be the kth term of $\mathcal{L}(a_k)$ and similarly with subscripts.

Conjecture

The sequence $\binom{n}{k}_{n\geq 0}$ *is infinitely log-concave for all* $k \geq 0$.

Proposition

We have

1. $\binom{n}{k}_{n\geq 0}$ *is infinitely log-concave for all* $k \leq 2$,

With more than one parameter, subscript $\mathcal{L}$ with the one which is varying.

Example. $\mathcal{L}_k \binom{n}{k}$ is for rows, $\mathcal{L}_n \binom{n}{k}$ is for columns.

Also, let $L(a_k)$ be the kth term of $\mathcal{L}(a_k)$ and similarly with subscripts.

Conjecture

The sequence $\binom{n}{k}_{n \geq 0}$ *is infinitely log-concave for all* $k \geq 0$.

Proposition

We have

1. $\binom{n}{k}_{n \geq 0}$ *is infinitely log-concave for all* $k \leq 2$,
2. $\mathcal{L}_n^i \binom{n}{k}$ *is nonnegative for all* k *and for* $i \leq 4$. □

Outline

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$.

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\left[\begin{array}{c} n \\ k \end{array} \right] = \dfrac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2] \cdots [n]$.

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\left[\begin{array}{c} n \\ k \end{array} \right] = \dfrac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2]\cdots[n]$.

Theorem

Let $n \geq 2$ and $k = \lfloor n/2 \rfloor$. Then

$$L_k^2\left(\left[\begin{array}{c} n \\ k \end{array} \right]\right) = -q^{n-2} + \text{ higher terms.} \qquad \square$$

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2] \cdots [n]$.

Theorem

Let $n \geq 2$ and $k = \lfloor n/2 \rfloor$. Then

$$L_k^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = -q^{n-2} + \text{ higher terms.} \quad \square$$

Conjecture

The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n \geq 0}$ is infinitely q-log-concave for all $k \geq 0$.

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2]\cdots[n]$.

Theorem

Let $n \geq 2$ and $k = \lfloor n/2 \rfloor$. Then

$$L_k^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = -q^{n-2} + \text{ higher terms.} \quad \square$$

Conjecture

The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n \geq 0}$ is infinitely q-log-concave for all $k \geq 0$.

Let $\langle n \rangle = q^{1-n} + q^{3-n} + \cdots + q^{n-1}$

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\left[\begin{matrix} n \\ k \end{matrix} \right] = \frac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2] \cdots [n]$.

Theorem

Let $n \geq 2$ and $k = \lfloor n/2 \rfloor$. Then

$$L_k^2 \left(\left[\begin{matrix} n \\ k \end{matrix} \right] \right) = -q^{n-2} + \text{ higher terms.} \quad \square$$

Conjecture

The sequence $\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]\right)_{n \geq 0}$ is infinitely q-log-concave for all $k \geq 0$.

Let $\langle n \rangle = q^{1-n} + q^{3-n} + \cdots + q^{n-1}$ and $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \frac{\langle n \rangle!}{\langle k \rangle! \langle n-k \rangle!}$

where $\langle n \rangle! = \langle 1 \rangle \langle 2 \rangle \cdots \langle n \rangle$.

A polynomial $f(q)$ is *q-nonnegative* if $f(q) \in \mathbb{R}_{\geq 0}[q]$. Call $(f_k(q))$ *infinitely q-log-concave* if $\mathcal{L}^i(f_k(q))$ is q-nonnegative for $i \geq 0$.

Let $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $\left[\begin{array}{c} n \\ k \end{array} \right] = \dfrac{[n]!}{[k]![n-k]!}$

where $[n]! = [1][2] \cdots [n]$.

Theorem

Let $n \geq 2$ and $k = \lfloor n/2 \rfloor$. Then

$$L_k^2\left(\left[\begin{array}{c} n \\ k \end{array} \right] \right) = -q^{n-2} + \text{ higher terms. } \quad \square$$

Conjecture

The sequence $\left(\left[{n \atop k} \right]\right)_{n \geq 0}$ is infinitely q-log-concave for all $k \geq 0$.

Let $\langle n \rangle = q^{1-n} + q^{3-n} + \cdots + q^{n-1}$ and $\left\langle {n \atop k} \right\rangle = \dfrac{\langle n \rangle!}{\langle k \rangle! \langle n-k \rangle!}$

where $\langle n \rangle! = \langle 1 \rangle \langle 2 \rangle \cdots \langle n \rangle$.

Conjecture

Sequences $\left(\left\langle {n \atop k} \right\rangle\right)_{k \geq 0}$ and $\left(\left\langle {n \atop k} \right\rangle\right)_{n \geq 0}$ are infinitely q-log-concave.

Outline

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$h_k(1^n) = \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition}$$

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$\begin{aligned} h_k(1^n) &= \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition} \\ &= \binom{n+k-1}{n-1}. \end{aligned}$$

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$\begin{aligned} h_k(1^n) &= \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition} \\ &= \binom{n+k-1}{n-1}. \end{aligned}$$

It follows that $(h_k(1^n))_{k \geq 0}$ is a column of Pascal's triangle

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$\begin{aligned} h_k(1^n) &= \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition} \\ &= \binom{n+k-1}{n-1}. \end{aligned}$$

It follows that $(h_k(1^n))_{k \geq 0}$ is a column of Pascal's triangle and $(h_k(1^{n-k}))_{k \geq 0}$ is a row.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$\begin{aligned} h_k(1^n) &= \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition} \\ &= \binom{n+k-1}{n-1}. \end{aligned}$$

It follows that $(h_k(1^n))_{k\geq 0}$ is a column of Pascal's triangle and $(h_k(1^{n-k}))_{k\geq 0}$ is a row. Define **x**-*nonnegativity* analogously to q-nonnegativity.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables. The *kth complete homogeneous symmetric function* is

$$h_k = h_k(\mathbf{x}) = \text{sum of all terms of degree } k \text{ in the } x_i.$$

Example. $h_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

Let 1^n denote the substitution

$$x_1 = x_2 = \ldots = x_n = 1, \quad x_{n+1} = x_{n+2} = \ldots = 0.$$

So

$$\begin{aligned} h_k(1^n) &= \#\text{ of ways to choose } k \text{ of } x_1, \ldots, x_n \text{ with repetition} \\ &= \binom{n+k-1}{n-1}. \end{aligned}$$

It follows that $(h_k(1^n))_{k\geq 0}$ is a column of Pascal's triangle and $(h_k(1^{n-k}))_{k\geq 0}$ is a row. Define **x**-*nonnegativity* analogously to q-nonnegativity.

Theorem

$\mathcal{L}^i(h_k(\mathbf{x}))$ is $\mathbf{x}$-nonnegative for $i \leq 3$ but not for $i = 4$.

Outline

The Boros-Moll Conjecture

Columns

q-analogues

Symmetric functions

Real roots

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

Suppose (a_k) is nonnegative. It is well known that

$$p[a_k] \text{ has only real roots} \implies (a_k) \text{ is log concave.}$$

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

Suppose (a_k) is nonnegative. It is well known that

$$p[a_k] \text{ has only real roots} \implies (a_k) \text{ is log concave.}$$

Example. If $(a_k) = \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

Suppose (a_k) is nonnegative. It is well known that

$$p[a_k] \text{ has only real roots } \implies (a_k) \text{ is log concave.}$$

Example. If $(a_k) = \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ then $p[a_k] = (1+x)^n$ has only real roots.

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

Suppose (a_k) is nonnegative. It is well known that

$$p[a_k] \text{ has only real roots} \implies (a_k) \text{ is log concave.}$$

Example. If $(a_k) = \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ then $p[a_k] = (1+x)^n$ has only real roots. It follows that $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.

If $(a_k) = a_0, a_1, \ldots, a_n$ is finite then let

$$p[a_k] = a_0 + a_1 x + \cdots + a_n x^n.$$

Suppose (a_k) is nonnegative. It is well known that

$$p[a_k] \text{ has only real roots} \implies (a_k) \text{ is log concave.}$$

Example. If $(a_k) = \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ then $p[a_k] = (1+x)^n$ has only real roots. It follows that $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.

Conjecture (M-S, Stanley, Fisk)

Suppose $(a_k) = a_0, a_1, \ldots, a_n$ is nonnegative. Then

$$p[a_k] \text{ has only real roots} \implies p[\mathcal{L}(a_k)] \text{ has only real roots.}$$