# The fractal nature of the Fibonomial triangle

Xi Chen
School of Mathematical Sciences, Dalian University of Technology,
Dalian City, Liaoning Province, 116024, P. R. China
xichen.dut@gmail.com

and

Bruce E. Sagan
Department of Mathematics, Michigan State University,
East Lansing, MI 48824-1027, USA
sagan@math.msu.edu
www.math.msu.edu/~sagan

October 11, 2013

# Outline

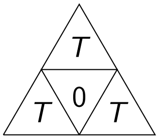
Here is Pascal's triangle modulo 2:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Here is Pascal's triangle modulo 2:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

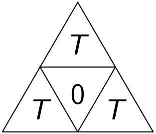
It is fractal: the triangle $T$ which is the first $2^m$ rows is repeated at the left and right of the next $2^m$ rows with 0s in between.



Here is Pascal's triangle modulo 2:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

It is fractal: the triangle $T$ which is the first $2^m$ rows is repeated at the left and right of the next $2^m$ rows with 0s in between.



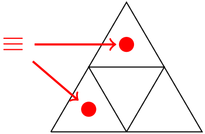

**Theorem**

*If $m \geq 0$ and $0 \leq k, n < 2^m$ then* $\binom{n+2^m}{k} \equiv \binom{n}{k} \pmod 2$.

Here is Pascal's triangle modulo 2:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

It is fractal: the triangle $T$ which is the first $2^m$ rows is repeated at the left and right of the next $2^m$ rows with 0s in between.



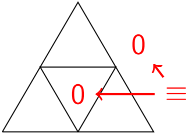

**Theorem**

*If $m \geq 0$ and $0 \leq k, n < 2^m$ then* $\binom{n+2^m}{k} \equiv \binom{n}{k} \pmod 2$.

Here is Pascal's triangle modulo 2:

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1

It is fractal: the triangle $T$ which is the first $2^m$ rows is repeated at the left and right of the next $2^m$ rows with 0s in between.



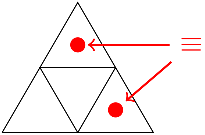

**Theorem**

*If $m \geq 0$ and $0 \leq k, n < 2^m$ then* $\binom{n+2^m}{k} \equiv \binom{n}{k} \pmod 2$.

Here is Pascal's triangle modulo 2:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  |  |  |  |  |  |  | 1 |  |  |  |  |  |  |  |
|  |  |  |  |  |  | 1 |  | 1 |  |  |  |  |  |  |
|  |  |  |  |  | 1 |  | 0 |  | 1 |  |  |  |  |  |
|  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  |  |  |  |
|  |  |  | 1 |  | 0 |  | 0 |  | 0 |  | 1 |  |  |  |
|  |  | 1 |  | 1 |  | 0 |  | 0 |  | 1 |  | 1 |  |  |
|  | 1 |  | 0 |  | 1 |  | 0 |  | 1 |  | 0 |  | 1 |  |
| 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |

It is fractal: the triangle $T$ which is the first $2^m$ rows is repeated at the left and right of the next $2^m$ rows with 0s in between.




### Theorem

*If $m \geq 0$ and $0 \leq k, n < 2^m$ then* $\binom{n+2^m}{k} \equiv \binom{n}{k} \pmod{2}$.

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $0 \leq k \leq n$ the *Fibonomials* are

$$\binom{n}{k}_F = \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_k F_1 F_2 \cdots F_{n-k}}.$$

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $0 \leq k \leq n$ the *Fibonomials* are

$$\binom{n}{k}_F = \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_k F_1 F_2 \cdots F_{n-k}}.$$

**Ex.** $\binom{6}{3}_F = \frac{F_1 F_2 \cdots F_6}{(F_1 F_2 F_3)^2} = \frac{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8}{(1 \cdot 1 \cdot 2)^2} = 60.$

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $0 \leq k \leq n$ the *Fibonomials* are

$$\binom{n}{k}_F = \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_k F_1 F_2 \cdots F_{n-k}}.$$

**Ex.** $\binom{6}{3}_F = \frac{F_1 F_2 \cdots F_6}{(F_1 F_2 F_3)^2} = \frac{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8}{(1 \cdot 1 \cdot 2)^2} = 60.$

The $\binom{n}{k}_F$ are always integers.

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $0 \leq k \leq n$ the *Fibonomials* are

$$\binom{n}{k}_F = \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_k F_1 F_2 \cdots F_{n-k}}.$$

**Ex.** $\binom{6}{3}_F = \frac{F_1 F_2 \cdots F_6}{(F_1 F_2 F_3)^2} = \frac{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8}{(1 \cdot 1 \cdot 2)^2} = 60.$

The $\binom{n}{k}_F$ are always integers. Modulo 2 we have:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 1 & & 1 & & & \\ & & 1 & & 0 & & 0 & & 1 & & \\ & 1 & & 1 & & 0 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $0 \leq k \leq n$ the *Fibonomials* are

$$\binom{n}{k}_F = \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_k F_1 F_2 \cdots F_{n-k}}.$$

**Ex.** $\binom{6}{3}_F = \frac{F_1 F_2 \cdots F_6}{(F_1 F_2 F_3)^2} = \frac{1 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 8}{(1 \cdot 1 \cdot 2)^2} = 60.$

The $\binom{n}{k}_F$ are always integers. Modulo 2 we have:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & 1 & & \\
 & 1 & & 1 & & 0 & & 1 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Theorem (Chen and S.)

*If $m \geq 0$ and $0 \leq k, n < 3 \cdot 2^m$ then*

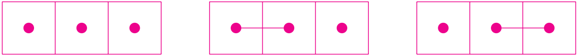
$$\binom{n + 3 \cdot 2^m}{k}_F \equiv \binom{n}{k}_F \pmod{2}.$$

# Outline

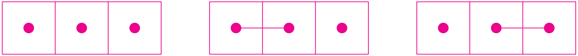
A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos.

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

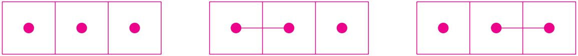
A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

Proposition

*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

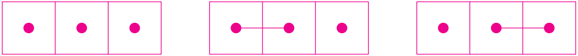
Proposition

*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

### Proposition

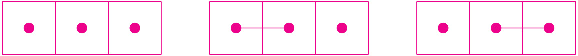
*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

### Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

Proof Sketch. We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively.

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

### Proposition

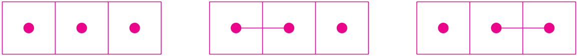
*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

### Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

Proof Sketch. We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively. If $T$ begins with a domino then $\iota(T)$ begins with 2 monominos and is otherwise the same.

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

## Proposition

*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

## Lemma

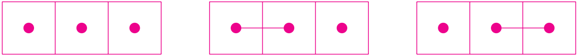
*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

Proof Sketch. We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively. If $T$ begins with a domino then $\iota(T)$ begins with 2 monominos and is otherwise the same.

**Ex.**

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$



### Proposition

*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

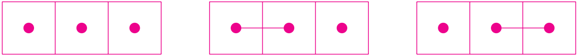
### Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

Proof Sketch. We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively. If $T$ begins with a domino then $\iota(T)$ begins with 2 monominos and is otherwise the same. Unpaired tilings begin with a monomino followed by a domino.

**Ex.**

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

## Proposition

*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

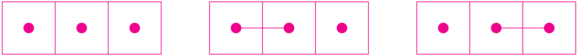
## Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

**Proof Sketch.** We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively. If $T$ begins with a domino then $\iota(T)$ begins with 2 monominos and is otherwise the same. Unpaired tilings begin with a monomino followed by a domino. Iterate, considering squares 4 and 5, etc. □

**Ex.**

A *tiling*, $T$, of a row of $n$ squares is a covering of the squares with disjoint dominos and monominos. Let $\mathcal{T}_n$ be the set of such.

**Ex.** $\mathcal{T}_3 =$ 

### Proposition

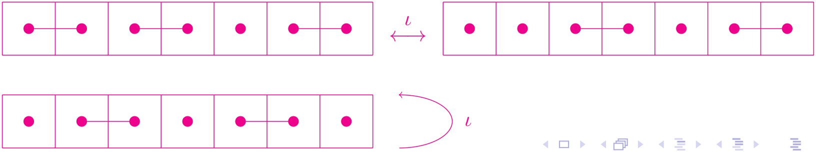
*For $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.*

### Lemma

*We have $F_n$ is even if and only if $n \equiv 0 \pmod 3$.*

**Proof Sketch.** We construct an involution $\iota$ on $\mathcal{T}_{n-1}$ with 0 or 1 fixed points if $n \equiv 0 \pmod 3$ or not, respectively. If $T$ begins with a domino then $\iota(T)$ begins with 2 monominos and is otherwise the same. Unpaired tilings begin with a monomino followed by a domino. Iterate, considering squares 4 and 5, etc. □

**Ex.**



S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions.

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

**Lemma**

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases}$$

□

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

### Lemma

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases} \quad \square$$

To finish the proof, use this lemma and the fractal nature of $\binom{n}{k}$.

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

### Lemma

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases}$$

□

To finish the proof, use this lemma and the fractal nature of $\binom{n}{k}$. For example, when in the third case of the lemma we have

$$\binom{n + 3 \cdot 2^m}{k}_F \equiv$$

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

**Lemma**

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases}$$

□

To finish the proof, use this lemma and the fractal nature of $\binom{n}{k}$. For example, when in the third case of the lemma we have

$$\binom{n+3\cdot 2^m}{k}_F \equiv \binom{\lfloor 2n/3 \rfloor + 2^{m+1}}{\lfloor 2k/3 \rfloor} \equiv$$

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

**Lemma**

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases}$$

□

To finish the proof, use this lemma and the fractal nature of $\binom{n}{k}$. For example, when in the third case of the lemma we have

$$\binom{n+3\cdot 2^m}{k}_F \equiv \binom{\lfloor 2n/3 \rfloor + 2^{m+1}}{\lfloor 2k/3 \rfloor} \equiv \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} \equiv$$

S and Savage have given a combinatorial interpretation for $\binom{n}{k}_F$ by tiling $k \times (n-k)$ rectangles containing partitions. One can extend the involution $\iota$ to such tilings.

### Lemma

*Let $f(n,k)$ be the number of fixed points of the action of $\iota$ so that $\binom{n}{k}_F \equiv f(n,k) \pmod 2$. Then*

$$f(n,k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \text{ and } k \equiv 1 \pmod 3, \\ \binom{\lceil 2n/3 \rceil}{\lceil 2k/3 \rceil} & \text{if } n \equiv 1 \pmod 3 \text{ and } k \equiv 0 \pmod 3, \\ \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} & \text{else.} \end{cases}$$

□

To finish the proof, use this lemma and the fractal nature of $\binom{n}{k}$. For example, when in the third case of the lemma we have

$$\binom{n+3\cdot 2^m}{k}_F \equiv \binom{\lfloor 2n/3 \rfloor + 2^{m+1}}{\lfloor 2k/3 \rfloor} \equiv \binom{\lfloor 2n/3 \rfloor}{\lfloor 2k/3 \rfloor} \equiv \binom{n}{k}_F \pmod 2.$$

□

# Outline

A *base* is an infinite sequence of integers $\mathbf{b} = (b_0 < b_1 < b_2 < \dots)$ with $b_0 = 1$ and $b_i | b_{i+1}$ for $i \geq 0$.

A *base* is an infinite sequence of integers $\mathbf{b} = (b_0 < b_1 < b_2 < \dots)$ with $b_0 = 1$ and $b_i | b_{i+1}$ for $i \geq 0$. In particular, consider

$$\mathbf{p} = (1, p, p^2, p^3, \dots) \quad \text{and} \quad \mathbf{F} = (1, 3, 3 \cdot 2, 3 \cdot 2^2, \dots).$$

A *base* is an infinite sequence of integers $\mathbf{b} = (b_0 < b_1 < b_2 < \dots)$ with $b_0 = 1$ and $b_i | b_{i+1}$ for $i \geq 0$. In particular, consider

$$\mathbf{p} = (1, p, p^2, p^3, \dots) \quad \text{and} \quad \mathbf{F} = (1, 3, 3 \cdot 2, 3 \cdot 2^2, \dots).$$

The *expansion* of $n \in \mathbb{Z}$ in base $\mathbf{b}$ is

$$n = n_0 b_0 + n_1 b_1 + n_2 b_2 + \cdots \stackrel{\text{def}}{=} (n_0, n_1, n_2, \dots)_{\mathbf{b}}$$

where $0 \leq n_i < b_{i+1}/b_i$ for all $i$.

A *base* is an infinite sequence of integers $\mathbf{b} = (b_0 < b_1 < b_2 < \dots)$ with $b_0 = 1$ and $b_i | b_{i+1}$ for $i \geq 0$. In particular, consider

$$\mathbf{p} = (1, p, p^2, p^3, \dots) \quad \text{and} \quad \mathbf{F} = (1, 3, 3 \cdot 2, 3 \cdot 2^2, \dots).$$

The *expansion* of $n \in \mathbb{Z}$ in base $\mathbf{b}$ is

$$n = n_0 b_0 + n_1 b_1 + n_2 b_2 + \cdots \stackrel{\text{def}}{=} (n_0, n_1, n_2, \dots)_{\mathbf{b}}$$

where $0 \leq n_i < b_{i+1}/b_i$ for all $i$.

**Theorem (Lucas)**

*Let $p$ be prime and $n = (n_0, n_1, \dots)_{\mathbf{p}}$, $k = (k_0, k_1, \dots)_{\mathbf{p}}$. Then*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2} \cdots \pmod{p}. \qquad \square$$

A *base* is an infinite sequence of integers $\mathbf{b} = (b_0 < b_1 < b_2 < \dots)$ with $b_0 = 1$ and $b_i | b_{i+1}$ for $i \geq 0$. In particular, consider

$$\mathbf{p} = (1, p, p^2, p^3, \dots) \quad \text{and} \quad \mathbf{F} = (1, 3, 3 \cdot 2, 3 \cdot 2^2, \dots).$$

The *expansion* of $n \in \mathbb{Z}$ in base $\mathbf{b}$ is

$$n = n_0 b_0 + n_1 b_1 + n_2 b_2 + \cdots \stackrel{\text{def}}{=} (n_0, n_1, n_2, \dots)_{\mathbf{b}}$$

where $0 \leq n_i < b_{i+1}/b_i$ for all $i$.

**Theorem (Lucas)**

*Let $p$ be prime and $n = (n_0, n_1, \dots)_{\mathbf{p}}$, $k = (k_0, k_1, \dots)_{\mathbf{p}}$. Then*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2} \cdots \pmod{p}. \qquad \square$$

**Theorem (Chen and S)**

*Let $n = (n_0, n_1, \dots)_{\mathbf{F}}$ and $k = (k_0, k_1, \dots)_{\mathbf{F}}$. Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod{2}. \qquad \square$$

## Theorem (Chen and S)

*Let* $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ *and* $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. *Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod 2. \qquad \square$$

## Theorem (Chen and S)

*Let* $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ *and* $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. *Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod{2}. \qquad \square$$

To finish the proof of the main theorem, if $0 \leq n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

## Theorem (Chen and S)

*Let* $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ *and* $k = (k_0, k_1, \ldots)_{\mathbf{F}}$*. Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod{2}. \qquad \square$$

To finish the proof of the main theorem, if $0 \le n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

Also

$$n + 3 \cdot 2^m = (n_0, \ldots, n_m, 1)_{\mathbf{F}}.$$

## Theorem (Chen and S)

*Let* $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ *and* $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. *Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod 2. \qquad \square$$

To finish the proof of the main theorem, if $0 \le n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

Also

$$n + 3 \cdot 2^m = (n_0, \ldots, n_m, 1)_{\mathbf{F}}.$$

$$\binom{n + 3 \cdot 2^m}{k}_F \equiv$$

### Theorem (Chen and S)

*Let* $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ *and* $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. *Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod{2}. \qquad \square$$

To finish the proof of the main theorem, if $0 \leq n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

Also

$$n + 3 \cdot 2^m = (n_0, \ldots, n_m, 1)_{\mathbf{F}}.$$

$$\binom{n + 3 \cdot 2^m}{k}_F \equiv \binom{n_0}{k_0}_F \cdots \binom{n_m}{k_m}_F \binom{1}{0}_F \pmod{2}$$

## Theorem (Chen and S)

*Let $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ and $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod{2}. \qquad \square$$

To finish the proof of the main theorem, if $0 \leq n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

Also

$$n + 3 \cdot 2^m = (n_0, \ldots, n_m, 1)_{\mathbf{F}}.$$

$$\begin{aligned} \binom{n + 3 \cdot 2^m}{k}_F &\equiv \binom{n_0}{k_0}_F \cdots \binom{n_m}{k_m}_F \binom{1}{0}_F \pmod{2} \\ &= \binom{n_0}{k_0}_F \cdots \binom{n_m}{k_m}_F \end{aligned}$$

## Theorem (Chen and S)

*Let $n = (n_0, n_1, \ldots)_{\mathbf{F}}$ and $k = (k_0, k_1, \ldots)_{\mathbf{F}}$. Then*

$$\binom{n}{k}_F \equiv \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \pmod 2. \qquad \square$$

To finish the proof of the main theorem, if $0 \leq n, k < 3 \cdot 2^m$ then

$$n = (n_0, \ldots, n_m)_{\mathbf{F}} \quad \text{and} \quad k = (k_0, \ldots, k_m)_{\mathbf{F}}.$$

Also

$$n + 3 \cdot 2^m = (n_0, \ldots, n_m, 1)_{\mathbf{F}}.$$

$$\begin{aligned} \binom{n + 3 \cdot 2^m}{k}_F &\equiv \binom{n_0}{k_0}_F \cdots \binom{n_m}{k_m}_F \binom{1}{0}_F \pmod 2 \\ &= \binom{n_0}{k_0}_F \cdots \binom{n_m}{k_m}_F \\ &\equiv \binom{n}{k}_F \pmod 2. \qquad \square \end{aligned}$$

# Outline

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$,

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square$$

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$*,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square$$

To finish the proof of the main theorem when $0 < n, k < 3 \cdot 2^m$, recall that the $F_n$ have period three modulo two.

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$*,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F.$$

□

To finish the proof of the main theorem when $0 < n, k < 3 \cdot 2^m$, recall that the $F_n$ have period three modulo two. So

$$\binom{n+3\cdot 2^m}{k}_F$$

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$,

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square$$

To finish the proof of the main theorem when $0 < n, k < 3 \cdot 2^m$, recall that the $F_n$ have period three modulo two. So

$$\binom{n+3\cdot 2^m}{k}_F = F_{n-k+3\cdot 2^m+1}\binom{n+3\cdot 2^m-1}{k-1}_F + F_{k-1}\binom{n+3\cdot 2^m-1}{k}_F$$

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$*,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square$$

To finish the proof of the main theorem when $0 < n, k < 3 \cdot 2^m$, recall that the $F_n$ have period three modulo two. So

$$\begin{aligned}\binom{n+3\cdot 2^m}{k}_F &= F_{n-k+3\cdot 2^m+1}\binom{n+3\cdot 2^m-1}{k-1}_F + F_{k-1}\binom{n+3\cdot 2^m-1}{k}_F \\ &\equiv F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F \pmod 2\end{aligned}$$

## Proposition

*We have* $\binom{0}{0}_F = 1$ *and, for* $n \geq 1$,

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square$$

To finish the proof of the main theorem when $0 < n, k < 3 \cdot 2^m$, recall that the $F_n$ have period three modulo two. So

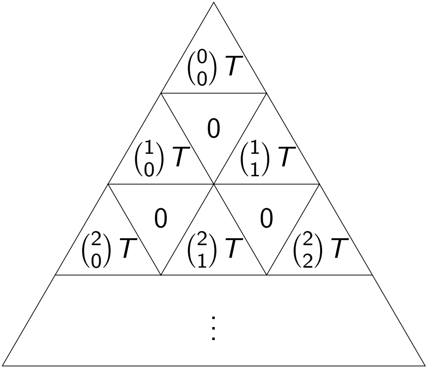
$$\begin{aligned}
\binom{n + 3 \cdot 2^m}{k}_F &= F_{n-k+3\cdot 2^m+1}\binom{n+3\cdot 2^m - 1}{k-1}_F + F_{k-1}\binom{n+3\cdot 2^m-1}{k}_F \\
&\equiv F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F \pmod 2 \\
&= \binom{n}{k}_F. \qquad \square
\end{aligned}$$

# Outline

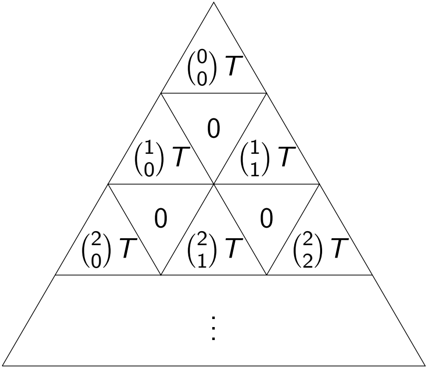
# 1. Other moduli.

**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$.

**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$. If $T$ is the triangle consisting of the first $p^m$ rows then the first $p^{m+1}$ rows are

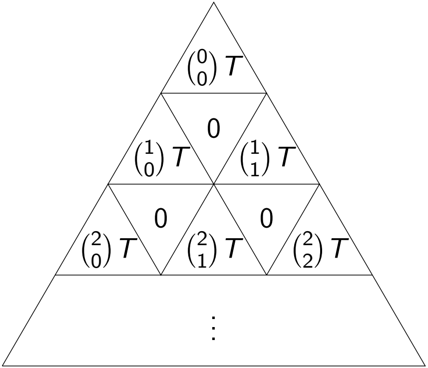


**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$. If $T$ is the triangle consisting of the first $p^m$ rows then the first $p^{m+1}$ rows are



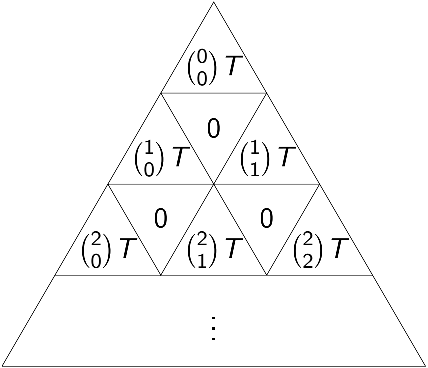

For $p = 3$, the Fibonomial triangle can be described in terms of triangles of side $4 \cdot 3^m$.

**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$. If $T$ is the triangle consisting of the first $p^m$ rows then the first $p^{m+1}$ rows are



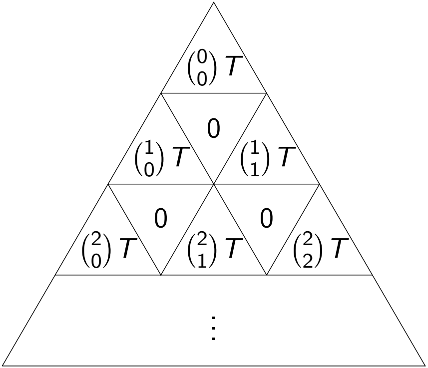

For $p = 3$, the Fibonomial triangle can be described in terms of triangles of side $4 \cdot 3^m$. But the result is much more complicated and we only have an inductive proof.

**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$. If $T$ is the triangle consisting of the first $p^m$ rows then the first $p^{m+1}$ rows are




For $p = 3$, the Fibonomial triangle can be described in terms of triangles of side $4 \cdot 3^m$. But the result is much more complicated and we only have an inductive proof. What can be said for other $p$?

**1. Other moduli.** Pascal's triangle is fractal modulo $p$ for any prime $p$. If $T$ is the triangle consisting of the first $p^m$ rows then the first $p^{m+1}$ rows are




For $p = 3$, the Fibonomial triangle can be described in terms of triangles of side $4 \cdot 3^m$. But the result is much more complicated and we only have an inductive proof. What can be said for other $p$? Note that it is an open problem to determine the period of the seqence $F_n$, $n \geq 0$, modulo $p$.

## 2. Lucas' congruence.

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3.

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3. If one considers the base

$$\mathbf{T} = (1, 4, 4 \cdot 3, 4 \cdot 3^2, \ldots)$$

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3. If one considers the base

$$\mathbf{T} = (1, 4, 4 \cdot 3, 4 \cdot 3^2, \ldots)$$

and lets $n = (n_0, n_1, \ldots)_{\mathbf{T}}$ and $k = (k_0, k_1, \ldots)_{\mathbf{T}}$ then it appears that

$$\binom{n}{k}_F \equiv 0 \pmod 3 \iff \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \equiv 0 \pmod 3.$$

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3. If one considers the base

$$\mathbf{T} = (1, 4, 4 \cdot 3, 4 \cdot 3^2, \ldots)$$

and lets $n = (n_0, n_1, \ldots)_{\mathbf{T}}$ and $k = (k_0, k_1, \ldots)_{\mathbf{T}}$ then it appears that

$$\binom{n}{k}_F \equiv 0 \pmod 3 \iff \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \equiv 0 \pmod 3.$$

But we do not always have equality of the Fibonomial above and the product.

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3. If one considers the base

$$\mathbf{T} = (1, 4, 4 \cdot 3, 4 \cdot 3^2, \dots)$$

and lets $n = (n_0, n_1, \dots)_{\mathbf{T}}$ and $k = (k_0, k_1, \dots)_{\mathbf{T}}$ then it appears that

$$\binom{n}{k}_F \equiv 0 \pmod 3 \iff \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \equiv 0 \pmod 3.$$

But we do not always have equality of the Fibonomial above and the product. Knuth and Wilf proved an analogue for Fibonomials of Kummer's Theorem describing the highest power of $p$ dividing a binomial coefficient.

**2. Lucas' congruence.** We were unable to find a Fibonomial analogue of Lucas' congruence modulo 3. If one considers the base

$$\mathbf{T} = (1, 4, 4 \cdot 3, 4 \cdot 3^2, \ldots)$$

and lets $n = (n_0, n_1, \ldots)_{\mathbf{T}}$ and $k = (k_0, k_1, \ldots)_{\mathbf{T}}$ then it appears that

$$\binom{n}{k}_F \equiv 0 \pmod 3 \iff \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \binom{n_2}{k_2}_F \cdots \equiv 0 \pmod 3.$$

But we do not always have equality of the Fibonomial above and the product. Knuth and Wilf proved an analogue for Fibonomials of Kummer's Theorem describing the highest power of $p$ dividing a binomial coefficient. This can be used to prove the Fibonomial analogue of Lucas' congruence modulo 2, but does not give enough information to do the same modulo 3.

# 3. Catalan numbers.

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F .$$

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F .$$

One can show that $C_{n,F} \in \mathbb{Z}$ for all $n$.

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F .$$

One can show that $C_{n,F} \in \mathbb{Z}$ for all $n$. For a base $\mathbf{b}$ and $n \in \mathbb{Z}$, let

$$\zeta_{\mathbf{b}}(n) = \text{number of nonzero digits of } (n)_{\mathbf{b}}.$$

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F.$$

One can show that $C_{n,F} \in \mathbb{Z}$ for all $n$. For a base $\mathbf{b}$ and $n \in \mathbb{Z}$, let

$$\zeta_{\mathbf{b}}(n) = \text{number of nonzero digits of } (n)_{\mathbf{b}}.$$

Theorem

*The 2-adic valuation of $C_n$ is*

$$\nu_2(C_n) = \zeta_2(n+1) - 1. \qquad \square$$

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F.$$

One can show that $C_{n,F} \in \mathbb{Z}$ for all $n$. For a base $\mathbf{b}$ and $n \in \mathbb{Z}$, let

$$\zeta_{\mathbf{b}}(n) = \text{number of nonzero digits of } (n)_{\mathbf{b}}.$$

Theorem

*The 2-adic valuation of $C_n$ is*

$$\nu_2(C_n) = \zeta_2(n+1) - 1. \qquad \square$$

Theorem (Amdeberhan, Chen, Moll, S)

*The 2-adic valuation of $C_{n,F}$ is*

$$\nu_2(C_{n,F}) = \begin{cases} \zeta_{\mathbf{F}}(n+1) & \text{if } n \equiv 3 \text{ or } 4 \pmod 6, \\ \zeta_{\mathbf{F}}(n+1) - 1 & \text{else.} \end{cases} \qquad \square$$

**3. Catalan numbers.** Louis Shapiro suggested defining a Fibonacci analogue of the Catalan numbers by

$$C_{n,F} = \frac{1}{F_{n+1}} \binom{2n}{n}_F .$$

One can show that $C_{n,F} \in \mathbb{Z}$ for all $n$. For a base **b** and $n \in \mathbb{Z}$, let

$$\zeta_{\mathbf{b}}(n) = \text{number of nonzero digits of } (n)_{\mathbf{b}}.$$

Theorem

*The 2-adic valuation of $C_n$ is*

$$\nu_{\mathbf{2}}(C_n) = \zeta_{\mathbf{2}}(n+1) - 1. \qquad \square$$

Theorem (Amdeberhan, Chen, Moll, S)

*The 2-adic valuation of $C_{n,F}$ is*

$$\nu_2(C_{n,F}) = \begin{cases} \zeta_{\mathbf{F}}(n+1) & \text{if } n \equiv 3 \text{ or } 4 \pmod 6, \\ \zeta_{\mathbf{F}}(n+1) - 1 & \text{else.} \end{cases} \qquad \square$$

Shapiro also asked for a combinatorial interpretation of $C_{n,F}$ but none is known.

THANKS FOR LISTENING!