Factoring the Characteristic Polynomial

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All posets P will be finite and have a unique minimal element $\hat{0}$ All P will also be *ranked* meaning that for every $x \in P$, all saturated $\hat{0}$–x chains will have the same length, $\rho(x)$. We also define the *rank of P* to be

$$\rho(P) = \max_{x \in P} \rho(x).$$

If μ is the Möbius function of P then the *characteristic polynomial* of P is

$$\chi(P) = \chi(P; t) = \sum_{x \in P} \mu(x) t^{\rho(P) - \rho(x)}.$$

Many ranked posets have characteristic polynomials whose roots are nonnegative integers. Why? Reasons have been given by Saito and Terao, Stanley, Zaslavsky, Blass and S, and others.

Proposition

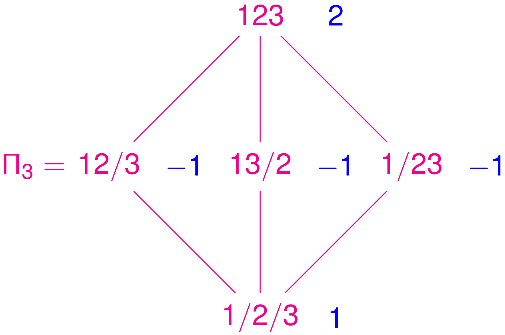
Let P, Q be ranked posets.
1. $P \cong Q \implies \chi(P; t) = \chi(Q; t)$.
2. $P \times Q$ is ranked and $\chi(P \times Q; t) = \chi(P; t)\chi(Q; t)$.

Let Π_n be the lattice of set partitions of $[n] = \{1, \ldots, n\}$ ordered by refinement.

Theorem

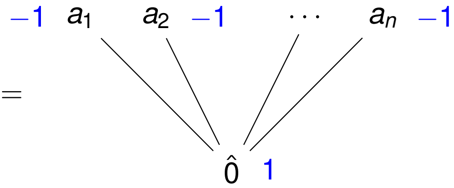
$\chi(\Pi_n, t) = (t-1)(t-2)\cdots(t-n+1).$

Ex. Consider Π_3.



$$\begin{aligned}\chi(\Pi_3, t) &= t^2 - t - t - t + 2\\ &= t^2 - 3t + 2\\ &= (t-1)(t-2).\end{aligned}$$

The *claw*, CL_n, consists of a $\hat{0}$ together with n atoms.

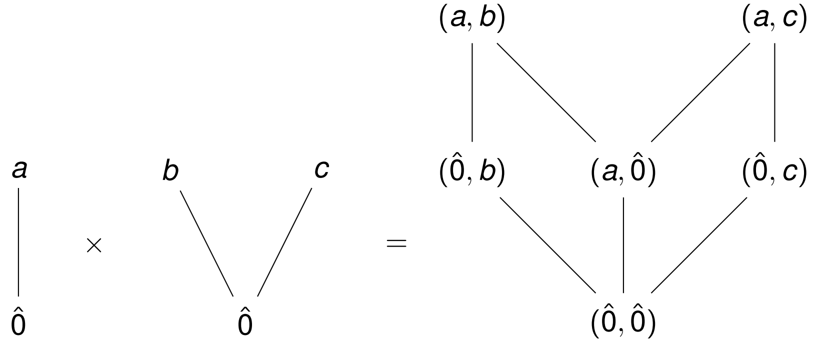
$CL_n =$ 

Thus

$$\chi(CL_n) = t - n.$$

So the characteristic polynomial of CL_n can give us any positive integer root as n varies.

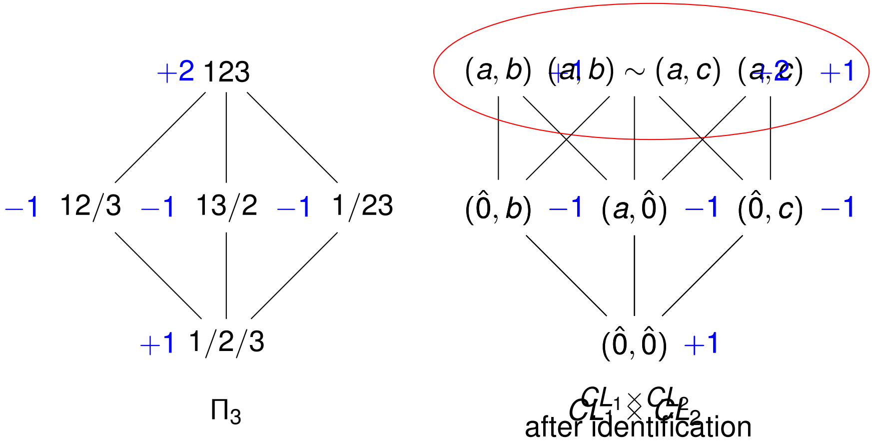
Let us consider the product $CL_1 \times CL_2$.



We have

$$\chi(CL_1 \times CL_2) = \chi(CL_1)\chi(CL_2) = (t-1)(t-2) = \chi(\Pi_3).$$

Clearly Π_3 and $CL_1 \times CL_2$ are not isomorphic. What if we identify the top two elements of $CL_1 \times CL_2$?



Note that the Möbius values of (a,b) and (a,c) added to give the Möbius value of $(a,b) \sim (a,c)$. So $\chi(CL_1 \times CL_2)$ did not change after the identification since characteristic polynomials only record the sums of the Möbius values at each rank.

General Method.

Suppose P is a ranked poset and we wish to prove

$$\chi(P) = (t - r_1) \dots (t - r_n)$$

where $r_1, \dots, r_n$ are positive integers.

1. Construct the poset

$$Q = CL_{r_1} \times \cdots \times CL_{r_n}.$$

2. Identify elements of Q to form a poset $Q/\sim$ in such a way that
   (a) $\chi(Q/\sim) = \chi(Q) = (t - r_1) \dots (t - r_n)$,
   (b) $(Q/\sim) \cong P$.

3. If follows that

$$\chi(P) = \chi(Q/\sim) = (t - r_1) \dots (t - r_n).$$

Let P be a poset and let $\sim$ be an equivalence relation on P. We define the *quotient*, $P/\sim$, to be the set of equivalence classes with the binary relation $\leq$ defined by

$$X \leq Y \text{ in } P/\sim \iff x \leq y \text{ in } P \text{ for some } x \in X \text{ and some } y \in Y.$$

Quotients of posets *need not* be posets.

Ex. Consider

$C_2 =$

Put an equivalence relation on C_2 with classes

$$X = \{0, 2\}, \qquad Y = \{1\}.$$

Then $X < Y$ since $0 < 1$ and $Y < X$ since $1 < 2$.

Let P be a poset and let $\sim$ be an equivalence relation on P. We say the quotient $P/\sim$ is a *homogeneous quotient* if

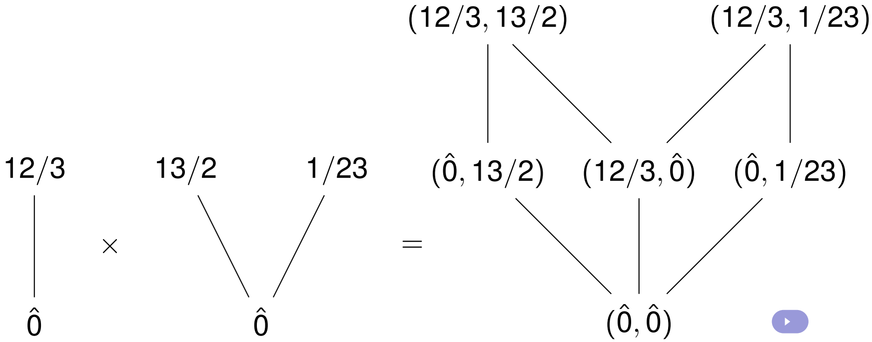
(1) $\hat{0}$ is in an equivalence class by itself, and
(2) $X \leq Y$ in $P/\sim$ implies that for all $x \in X$ there is a $y \in Y$ with $x \leq y$.

Lemma (Hallam-S)

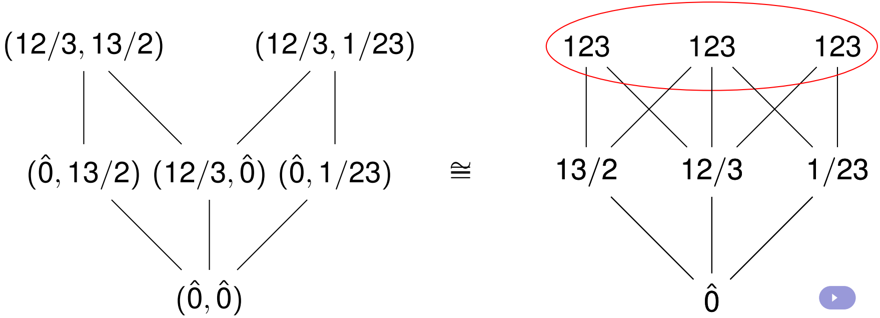
If $P/\sim$ is a homogeneous quotient then $P/\sim$ a poset.

How do we determine a suitable equivalence relation? If P is a lattice, then there is a canonical choice.

Let us revisit Π_3. Label the atoms of $CL_1 \times CL_2$ with atoms from Π_3 as follows:



Now relabel each element of the product with the join of its two coordinates.



Finally, identify elements with the same label to obtain the same quotient we did before. Not only is the quotient isomorphic to Π_3, it even has the same labeling.

An ordered partition of a set $\mathcal{A}$ is a sequence of subsets $(A_1, \ldots, A_n)$ with $\uplus_i A_i = \mathcal{A}$. We write $(A_1, \ldots, A_n) \vdash \mathcal{A}$.
Let $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$, where $\mathcal{A}(L)$ is the atom set of a lattice L. Let CL_{A_i} be the claw with atom set A_i. The *standard equivalence relation* on $\prod_i CL_{A_i}$ is

$$\mathbf{t} \sim \mathbf{s} \text{ in } \prod_{i=1}^{n} CL_{A_i} \iff \bigvee \mathbf{t} = \bigvee \mathbf{s} \text{ in } L.$$

The *atomic transversals of $x \in L$* are the elements of the equivalence class

$$\mathcal{T}_x^a = \left\{ \mathbf{t} \in \prod_{i=1}^{n} CL_{A_i} \ : \ \bigvee \mathbf{t} = x \right\}.$$

Ex. $(A_1, A_2) \vdash \mathcal{A}(\Pi_3)$ with $A_1 = \{12/3\}$, $A_2 = \{13/2, 1/23\}$.
Note that CL_{A_1} and CL_{A_2} were the claws used for Π_3.

We need a condition on the standard equivalence relation which will make sure that the quotient is homogeneous and ranked. The *support* of $\mathbf{t} = (t_1, \ldots, t_n) \in \prod_i CL_{A_i}$ is

$$\operatorname{supp} \mathbf{t} = \{i \ : \ t_i \neq \hat{0}\}.$$

Note that $|\operatorname{supp} \mathbf{t}| = \rho(\mathbf{t})$ where the rank is taken in $\prod_i CL_{A_i}$.

Lemma (Hallam-S)

Let L be a lattice, $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that for all $x \in L$ and all $\mathbf{t} \in \mathcal{T}_x^a$ we have

$$|\operatorname{supp} \mathbf{t}| = \rho(x).$$

Then the standard equivalence relation is homogeneous, $Q/\sim$ is ranked, and

$$\rho(\mathcal{T}_x^a) = \rho(x).$$

We wish to make sure that when identifying the elements in an equivalence class, the Möbius function of the class is the sum of the Möbius functions of its elements so that χ does not change. Given $x \in L$, let

$$A_x = \{a \in \mathcal{A}(L) \ : \ a \leq x\}.$$

Lemma (Hallam-S)

Let lattice L, $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ and $Q = \prod_i CL_{A_i}$ satisfy the conditions of the previous lemma. Suppose, for each $x \neq \hat{0}$ in L,

$$\text{there exists an index } i \text{ such that } |A_x \cap A_i| = 1. \tag{1}$$

Then for any $\mathcal{T}_x^a \in Q/\sim$ we have

$$\mu(\mathcal{T}_x^a) = \sum_{\mathbf{t} \in \mathcal{T}_x^a} \mu(\mathbf{t}).$$

Our main theorem is as follows.

Theorem (Hallam-S)

Let L be a lattice, $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that the following three conditions hold.

(1) *For all $x \in L$ we have $\mathcal{T}_x^a \neq \emptyset$.*

(2) *If $\mathbf{t} \in \mathcal{T}_x^a$ then $|\operatorname{supp} \mathbf{t}| = \rho(x)$.*

(3) *For each $x \neq \hat{0}$ in L, there is i such that $|A_x \cap A_i| = 1$.*

Then for the standard equivalence relation we can conclude the following.

(a) $(Q/\sim) \cong L$.

(b) $$\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|).$$

Condition (1) is used to prove that the map $(Q/\sim) \to L$ by $\mathcal{T}_x^a \mapsto x$ is surjective.

Corollary

$\chi(\Pi_n; t) = (t-1)(t-2)\dots(t-n+1).$

Proof. If $i < j$ let $\{i, j\}$ be the atom of Π_n having this set as its unique non-singleton block. Let $(A_1, \dots, A_{n-1}) \vdash \mathcal{A}(\Pi_n)$ where

$$A_i = \{\{1, i+1\}, \{2, i+1\}, \dots, \{i, i+1\}\}.$$

We will verify the three conditions for $x = \hat{1}$.

(1) $(\{1,2\}, \{2,3\}, \dots, \{n-1, n\}) \in \mathcal{T}^a_{\hat{1}}$.

(2) With any $\mathbf{t} \in Q$, associate a graph $G_\mathbf{t}$ with $V = [n]$ and

$$ij \in E \iff \{i, j\} \in \mathbf{t}.$$

I claim $G_\mathbf{t}$ is a forest. If $C : \dots i, m.j, \dots$ is a cycle with $m = \max C$, then $\{i, m\}, \{j, m\} \in \mathbf{t}$. But $\{i, m\}, \{j, m\} \in A_{m-1}$. Also, the vertices of the components of $G_\mathbf{t}$ are the blocks of $\bigvee \mathbf{t}$.

$$\therefore \mathbf{t} \in \mathcal{T}^a_{\hat{1}} \implies G_\mathbf{t} \text{ a tree} \implies |\operatorname{supp} \mathbf{t}| = n-1 = \rho(\hat{1}).$$

(3) $A_1 = \{\{1,2\}\}$ so $|A_{\hat{1}} \cap A_1| = 1$.

$\therefore \chi(\Pi_n; t) = (t - |A_1|) \dots (t - |A_{n-1}|) = (t-1)\dots(t-n+1)$. □

How do we find an appropriate atom partition? We say $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ is *induced by a chain* if there is a chain $C : \hat{0} = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}$ such that

$$A_i = \{a \in \mathcal{A}(L) \ : \ a \le x_i \text{ and } a \not\le x_{i-1}\}.$$

Ex. In Π_n, our partition is induced by $\hat{0} < [2] < [3] < \cdots < \hat{1}$ where $[i]$ is the partition having this set as its only non-trivial block.

When will the partition induced by such a chain give the roots of a factorization? For $x \in L$ with $x \neq \hat{0}$, let i be the index with $x \le x_i$ and $x \not\le x_{i-1}$. Say that C satisfies the *meet condition* if, for every $x \in L$ of rank at least 2,

$$x \wedge x_{i-1} \neq \hat{0}.$$

Our second main theorem is as follows.

Theorem (Hallam-S)

Let L be a lattice and $(A_1, \ldots, A_n)$ induced by a chain C. Suppose that for all $x \in L$ and $\mathbf{t} \in \mathcal{T}_x^a$ we have

$$|\operatorname{supp} \mathbf{t}| = \rho(x).$$

Under these conditions, the following are equivalent.

1. *For each $x \neq \hat{0}$ in L, there is i such that $|A_x \cap A_i| = 1$.*
2. *Chain C satisfies the meet condition.*
3. *The characteristic polynomial of L factors as*

$$\chi(L, t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|).$$

Any lattice L satisfies: for all $x, y, z \in L$ with $y \leq z$

$$y \vee (x \wedge z) \leq (y \vee x) \wedge z \quad (\textit{modular inequality}). \qquad (2)$$

Call $x \in L$ *left-modular* if, together with any $y \leq z$, we have equality in (2). A lattice is *supersolvable* if it has a saturated $\hat{0}$–$\hat{1}$ chain of left-modular elements.

Lemma (Hallam-S)

Let L be a lattice and C a $\hat{0}$–$\hat{1}$ chain in L inducing $(A_1, \ldots, A_n)$.

1. *If C is saturated and consists of left-modular elements, then C satisfies the meet condition.*
2. *If L is semimodular then for any $x \in L$ and $\mathbf{t} \in \mathcal{T}_x^a$ we have $|\operatorname{supp} \mathbf{t}| = \rho(x)$.*

Corollary (Stanley, 1972)

Let L be a semimodular, supersolvable lattice and $(A_1, \ldots, A_n)$ be induced by a saturated chain of left-modular elements. Then

$$\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|).$$