Descent and peak polynomials

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The cast of charactcers

SB = Sara Billey
KB = Krzysztof Burdzy
FCV = Francis Castro-Velez
ADL = Alexander Diaz-Lopez
MF = Matthew Fahrbach
PH = Pamela Harris
EI = Erik Insko
MO = Mohamed Omar
RO = Rosa Orellana
JP = José Pastrana
DPL = Darleen Perez-Lavin
BES = Bruce E Sagan
AT = Alan Talmage
RZ = Rita Zevallos

$$[n] := \{1, 2, \ldots, n\},$$
$$\mathfrak{S}_n := \text{symmetric group on } [n],$$
$$I_0 := I \cup \{0\} \text{ for } I \text{ a finite set of positive integers},$$
$$m := \max I_0.$$

Permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ has *descent set*

$$\operatorname{Des} \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1].$$

Given I and $n > m$, define

$$D(I; n) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des} \pi = I\} \quad \text{and} \quad d(I; n) = \#D(I; n).$$

Ex. $D(\{1, 2\}; 5) = \{32145, 42135, 52134, 43125, 53124, 54123\}$.

Theorem (MacMahon, 1916)

We have $d(I; n)$ is a polynomial in n, called the descent polynomial.

Proof. Let $I = \{i < j < \ldots\}$. Use inclusion-exclusion on $\pi \in \mathfrak{S}_n$ of the form $\pi = \pi_1 < \cdots < \pi_i \ \pi_{i+1} < \cdots < \pi_j \ \cdots$. □

Corollary (ADL-PH-EI-BES, 2016)

If $I \neq \emptyset$ and $I^- = I - \{m\}$ then $d(I; n) = \binom{n}{m} d(I^-; m) - d(I^-; n)$. So $\deg d(I; n) = m$.

$$[\ell, n] := [\ell, \ell+1, \ldots, n].$$

Permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ has *peak set*

$$\operatorname{Peak} \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1].$$

Note that if $\operatorname{Peak} \pi = I$ then I can not contain two consecutive integers and call such I *admissible*. If $n > m$ then define

$$P(I; n) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Peak} \pi = I\}.$$

Ex. $P(\{2\}; 4) = \{1324, 1423, 1432, 2314, 2413, 2431, 3412, 3421\}$.

Theorem (SB-KB-BES, 2013)

If $I \neq \emptyset$ is admissible then $\#P(I; n) = p(I; n) 2^{n-\#I-1}$ where $p(I; n)$ is a poynomial in n of degree $m-1$ called the peak polynomial.

Proof. Use inclusion-exclusion on $\pi \in \mathfrak{S}_n$ such that

$$\operatorname{Peak}(\pi_1 \ldots \pi_{m-1}) = I - \{m\} \quad \text{and} \quad \operatorname{Peak}(\pi_m \ldots \pi_n) = \emptyset$$

and then induct. □

The peak polynomial is not always real rooted. But it does have some interesting integral roots.

Theorem (SB-MF-AT, 2016)

Let $I = \{i_1 < \cdots < i_s\}$.

(i) *If* $i_{r+1} - i_r$ *is odd for some* r *then*

$$p(I; 0) = p(I; 1) = \cdots = p(I; i_r) = 0.$$

(ii) *If* $i \in I$ *then*

$$p(I; i) = 0.$$

In some ways the descent polynomial behaves similarly.

Theorem (ADL-PH-EI-BES, 2016)

If $i \in I$ then

$$d(I; i) = 0.$$

Proof.

$$d(I; n) = \binom{n}{m} d(I^-; m) - d(I^-; n)$$

where $I^- = I - \{m\}$. If $i < m$ then, using induction,

$$d(I; i) = \binom{i}{m} d(I^-; m) - d(I^-; i) = 0 \cdot d(I^-; m) - 0 = 0.$$

If $i = m$ then

$$d(I; m) = \binom{m}{m} d(I^-; m) - d(I^-; m) = 0$$

as desired. □

Ex. Let $I = \{1, 2\}$. Then

$$D(I; n) = \{\pi = \pi_1 > \pi_2 > \pi_3 < \pi_4 < \cdots < \pi_n\}.$$

So $\pi_3 = 1$. And picking any two elements of $[2, n]$ for π_1, π_2 determines π. Thus

$$d(I; n) = \binom{n-1}{2} = \frac{n^2 - 3n + 2}{2}$$

has negative, nonintegral coefficients.

The next peak polynomial result was conjectured by SB-KB-BES.

Theorem (ADL-PH-EI-MO, 2016)

The coefficients in the expansion

$$p(I; n) = \sum_{k \geq 0} a_k(I) \binom{n-m}{k}$$

are nonnegative integers.

Proof. Use a new recursion for $p(I; n)$ based on where $n + 1$ can be placed in passing from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$. □

For descent polynomials, these coefficients have a combinatorial interpretation.

Theorem (ADL-PH-EI-BES, 2016)

Define $b_k(I)$ as the coefficients in the expansion

$$d(I;n) = \sum_{k\geq 0} b_k(I)\binom{n-m}{k}.$$

Then $b_k(I)$ is the number of $\pi \in D(I;n)$ with

$$\{\pi_1 \ldots, \pi_m\} \cap [m+1, n] = [m+1, m+k]. \qquad (1)$$

Proof. Partition $D(I;n)$ into subsets $D_k(I;n)$ which contain those permutations in $D(I;n)$ such that $|\{\pi_1 \ldots, \pi_m\} \cap [m+1, n]| = k$. Then show

$$|D_k(I;n)| = b_k(I)\binom{n-m}{k}$$

where $b_k(I)$ is given by equation (1). □

More on roots (including complex).

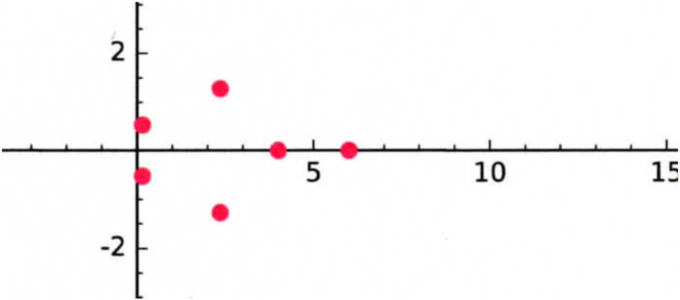
Conjecture (SB-MF-AT for p, ADL-PH-EI-BES for d, 2016)

If $d(I;z) = 0$, or if I is admissible and $p(I;z) = 0$ then

$$|z| \leq m \quad \textit{and} \quad \Re(z) \geq -3.$$

For $d(I;z)$ this conjecture has been checked for all I with $m \leq 12$.

Ex. Roots of $d(I;z)$ for $I = \{4, 6\}$.



More on coefficients.

Problem

Find a combinatorial interpretation of the $a_k(I)$ in

$$p(I;n) = \sum_{k\geq 0} a_k(I)\binom{n-m}{k}.$$

Sequence $a_0, a_1, \ldots$ is *log concave* if, for all k, $a_{k-1}a_{k+1} \leq a_k^2$.

Conjecture (ADL-PH-EI-BES, 2016)

The sequence $b_0(I), b_1(I), \ldots$ is log concave where the $b_k(I)$ are defined by

$$d(I;n) = \sum_{k\geq 0} b_k(I)\binom{n-m}{k}.$$

Note that the stronger condition of the generating function for $b_0(I), b_1(I), \ldots$ being real rooted does not always hold.

Proposition (ADL-PH-EI-BES, 2016)

If $I = [\ell, m]$ then $b_0(I), b_1(I), \ldots$ is log concave.

Other Coxeter groups.

The symmetric group is the Coxeter group of type A. There are analogous results for types B and D which have been demonstrated by FCV-ADL-RO-JP-RZ (2013) and ADL-PH-EI-DPL (2016) for $p(I; n)$, and by ADL-PH-EI-BES (2016) for $d(I; n)$. For example, we view $\beta = \beta_1 \ldots \beta_n \in B_n$ as a signed permutation and extend β to $\beta = \beta_0\beta_1 \ldots \beta_n$ where $\beta_0 = 0$. Translating the usual definition of descent set for a Coxeter system into this setting gives

$$\mathrm{Des}\,\beta = \{i \geq 0 \mid \beta_i > \beta_{i+1}\}.$$

Given a finite set I of nonnegative integers, define

$$D_B(I; n) = \{\beta \in B_n \mid \mathrm{Des}\,\beta = I\} \quad \text{and} \quad d_B(I; n) = \#D_B(I; n).$$

Using Inclusion-Exclusion, one obtains the following.

Proposition (ADL-PH-EI-BES, 2016)

If $I \neq \emptyset$ and $I^- = I - \{m\}$ then

$$d_B(I; n) = \binom{n}{m} 2^{n-m} d_B(I^-; m) - d_B(I^-; n).$$

THANKS FOR
LISTENING!