Discrete Morse Theory and Generalized Factor Order

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Outline

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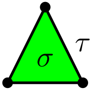

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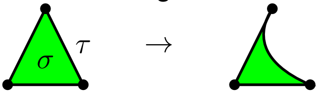

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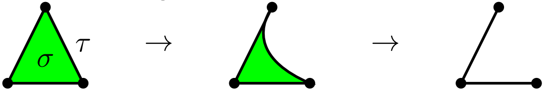

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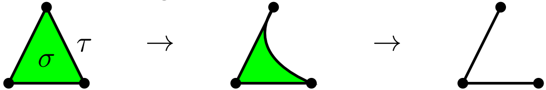

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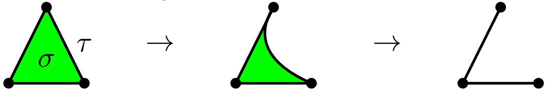

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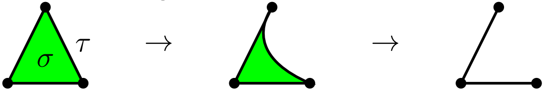

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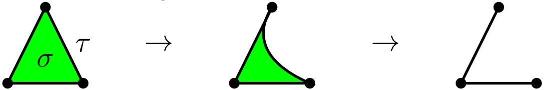

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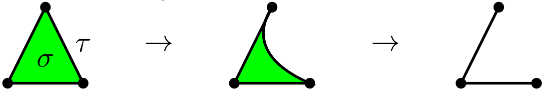

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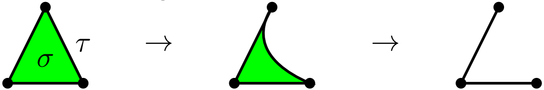

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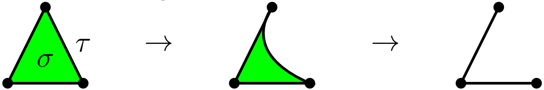

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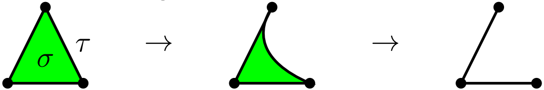

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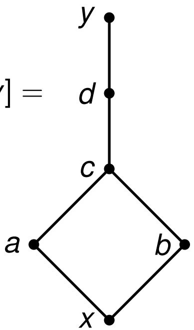

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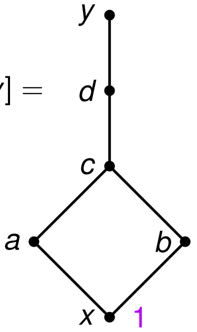

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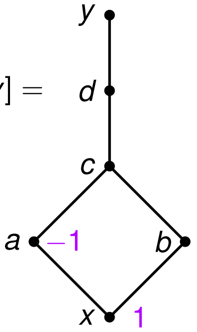

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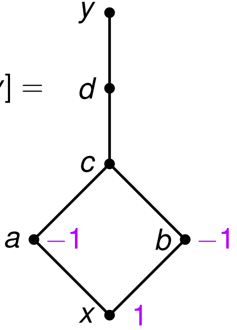
$[x, y] =$ 

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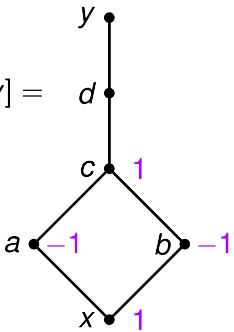

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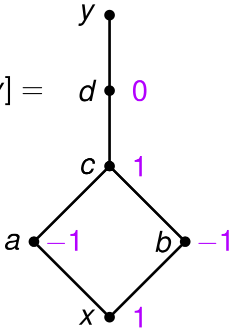

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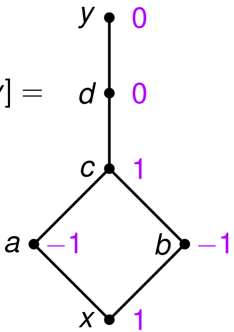

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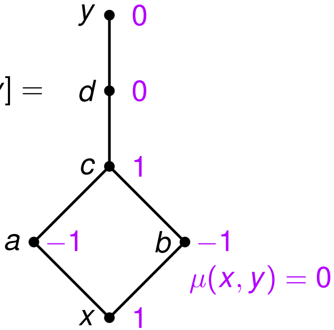

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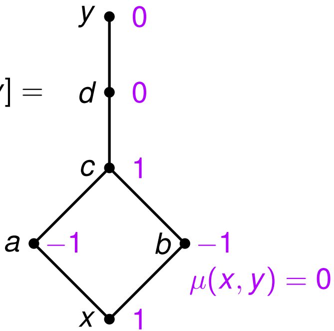

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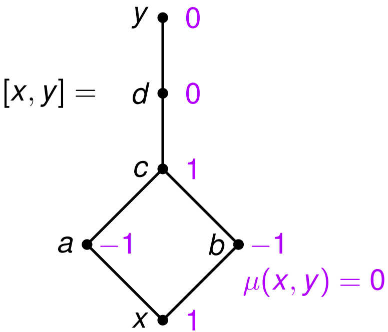

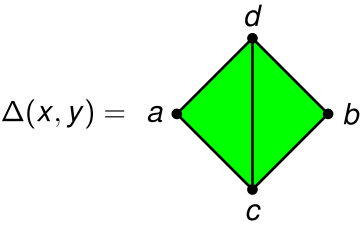

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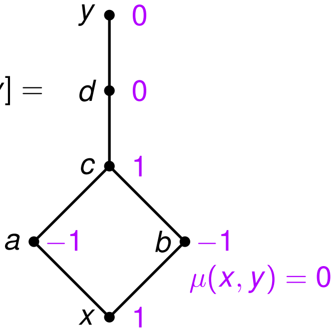

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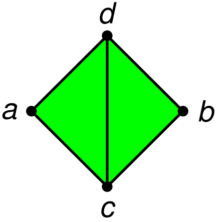

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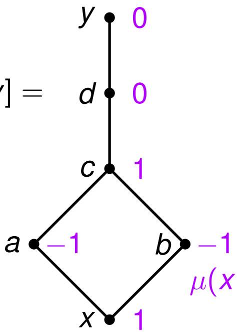

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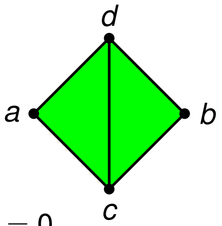

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$$\text{matching all new faces of each } C \text{ except perhaps one.} \qquad (1)$$

$$B : x = x_0, x_1, \ldots, x_m = y \quad \text{and} \quad C : x = y_0, y_1, \ldots, y_n = y$$

agree to level k if $x_0 = y_0, \ldots, x_k = y_k$. They *diverge from level k* if they agree to level k but $x_{k+1} \neq y_{k+1}$.

Ex. $B : x, a, c, d, y$ and $C : x, b, c, d, y$ diverge from level 0.

Call $\preceq$ a *poset lexicographic (PL) order* if, whenever C, D diverge from some level k and C', D' agree with C, D respectively to level $k + 1$, then

$$C \preceq D \iff C' \preceq D'.$$

Let $\mathcal{C} = \mathcal{C}(x, y)$ be the set of containment-maximal chains in (x, y). By convention, list the elements of $C \in \mathcal{C}$ from smallest to largest including x, y. If $\preceq$ is a total order on $\mathcal{C}$ then a *new face* of $C \in \mathcal{C}$ is $C' \subseteq C$ with $C' \not\subseteq B$ for all $B \prec C$.

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Proposition (Babson and Hersh)

If $\preceq$ is an PL-order then it has a MM satisfying (1).

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$$\mu(x, y) = \sum_C (-1)^{\#\mathcal{J}(C)-1}$$

where the sum is over all critical $C \in \mathcal{C}(x, y)$.

Outline

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\mu(u, w) = \begin{cases}
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Ex. $\mu(a, abbab)$

Let A be a set (the *alphabet*) and let A^* be the set of words w over A. Call $u \in A^*$ a *factor* of w if $w = xuy$ for some $x, y \in A^*$.

Ex. $u = abba$ is a factor of $w = baabbaa$.

Factor order on A^* is the partial order $u \leq w$ if u is a factor of w. The *inner* and *outer factors* of $w = a_1 a_2 \ldots a_n$ are

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Theorem (Björner)

In factor order on A^*

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So any maximal chain $C : w = w_0, w_1, \ldots, w_m = u$ determines a chain of embeddings with labels $l(C) = (l_1, \ldots, l_n)$

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Lemma (S and Willenbring)

The total order on $\mathcal{C}(w, u)$ given by $B \preceq C$ iff $l(B) \leq_{\text{lex}} l(C)$ is a PL-order

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Proof Let B be the chain which goes from w to the suffix embedding of $o(w)$ and then continues to u as does C. Then $B \prec C$ and $C - I \subseteq B$ so I is an SI. Because $o(w) \not\le i(w)$ there are only two embeddings of $o(w)$ in w: prefix and suffix. Thus, since C is lexicographically first through the prefix embedding, any other chain prior to C must agree with the portion of B up to the suffix embedding of $o(w)$.

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Outline

1. Babson, E., and Hersh, P., Discrete Morse functions from lexicographic orders, *Trans. Amer. Math. Soc. 357*, 2 (2005), 509–534 (electronic).
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THANKS FOR LISTENING!