Log-concavity and -convexity via distributive lattices

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Consider a sequence of real numbers

$$(a_n) = (a_n)_{n \geq 0} = a_0, a_1, a_2, \ldots$$

Say that (a_n) is *log-concave* if, for all $n \geq 1$, we have

$$a_n^2 \geq a_{n-1} a_{n+1}.$$

Say that (a_n) is *log-convex* if, for all $n \geq 1$, we have

$$a_n^2 \leq a_{n-1} a_{n+1}.$$

Log-concavity and -convexity have been intensely studied in combinatorics, algebra, and geometry.

A *lower order ideal* of poset $P = (P, \preceq)$ is $I \subseteq P$ such that

$$x \in I \text{ and } y \preceq x \text{ implies } y \in I.$$

An *upper order ideal* of P is $J \subseteq P$ such that

$$x \in J \text{ and } y \succeq x \text{ implies } y \in J.$$

Poset L is a *lattice* if evey $x, y \in L$ have a *meet* $x \wedge y$ (greatest lower bound), and a *join* $x \vee y$ (least upper bound). Say L is *distributive* if it satisfies one of the two equivalent distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in L,$$

or

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \in L.$$

The following follows easily from the FKG (Fortuin, Kasteleyn, Ginibre) inequality.

Lemma (The Order Ideal Lemma, OIL, Liang-S)

Let L be a distributive lattice and $I, J \subseteq L$ be ideals.

(a) *If I, J are both lower ideals or both upper ideals then*

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|$$

where $|\ |$ denotes cardinality.

(b) *If one of I, J is a lower order ideal and the other is upper then*

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|.$$

The Order Ideal Lemma can be used to prove log-concavity and log-convexity results about

1. Lattice paths including Dyck, Motzkin, Schröder
2. Integer partitions
3. Order polynomials including semistandard Young tableaux
4. Generalized Lucas sequences including the Fibonacci sequence
5. Descent and peak polynomials
6. Pattern avoidance classes
7. Stirling numbers of the second kind
8. Noncrossing partitions and Narayana numbers
9. q-analogues

The *Catalan numbers* are

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

So

$$(C_n) = 1, 1, 2, 5, 14, 42, \ldots$$

A *Dyck path of semilength n* is a lattice path P satisfying:

1. P starts at $(0, 0)$ and ends at $(2n, 0)$,
2. P uses up steps U parallel to the vector $[1, 1]$ and down steps D parallel to the vector $[1, -1]$,
3. never goes below the x-axis.

Let

$$\mathcal{D}_n = \{P \mid P \text{ is a Dyck path of semilength } n\}.$$

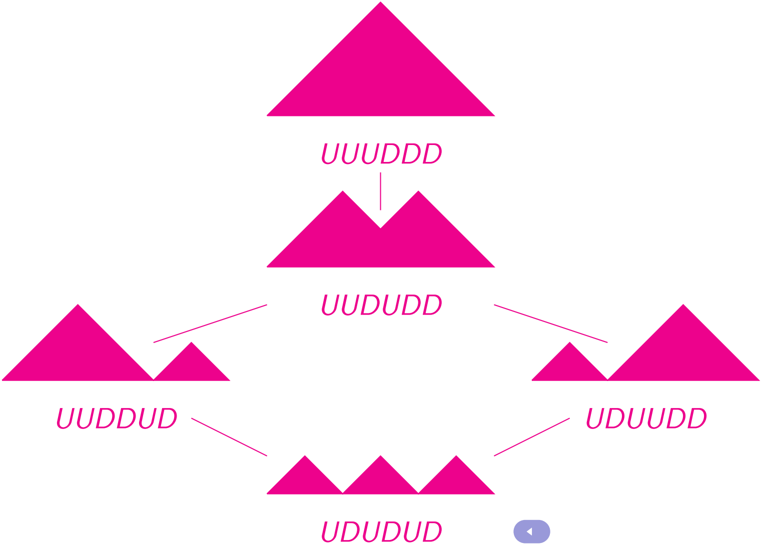
It is well known that

$$C_n = |\mathcal{D}_n|.$$

If $P \in \mathcal{D}_n$ then let $A(P)$ be the physical area enclosed by P and the x-axis. Turn $\mathcal{D}_n$ into a poset by letting

$$P \preceq Q \iff A(P) \subseteq A(Q).$$

Ex. $\mathcal{D}_3$



Lemma (Ferrari, Pinzani)

For all $n \geq 0$ we have that $\mathcal{D}_n$ is a distributive lattice. □

Theorem

The sequence (C_n) is log-convex.

Proof (Liang-S). Let $L = \mathcal{D}_{n+1}$ so that $|L| = C_{n+1}$. The maximal element in $\mathcal{D}_n$ is the path $U^n D^n$. Construct the lower order ideals

$$I = \{P \in L \mid P \preceq U^n D^n UD\}$$

and

$$J = \{P \in L \mid P \preceq UDU^n D^n\}.$$

Therefore $I \cong J \cong \mathcal{D}_n$ so that $|I| = |J| = C_n$. Also

$$I \cap J = \{P \in L \mid P \preceq UDU^{n-1} D^{n-1} UD\}.$$

Therefore $I \cap J \cong \mathcal{D}_{n-1}$ so that $|I \cap J| = C_{n-1}$. Thus, by the Order Ideal Lemma

$$C_n^2 = |I| \cdot |J| \leq |I \cap J| \cdot |L| = C_{n-1} C_{n+1}$$

as desired. □

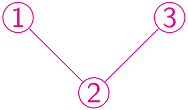
For a positive integer p we let $[p] = \{1, 2, \ldots, p\}$. Let $(P, \preceq)$ be a poset on $[p]$. A *P-partition with range $[n]$* is a map $f : P \to [n]$ such that for all $x \prec y$:

1. $f(x) \geq f(y)$, i.e., f is order reversing, and
2. If $x > y$ then $f(x) > f(y)$.

Let

$$\mathcal{O}_P(n) = \{f \mid f \text{ is a } P\text{-partition with range } [n]\}.$$

Ex. Suppose $p = 3$ and

$P =$ 

$$\therefore \mathcal{O}_P(n) = \{f : P \to [n] \mid f(2) > f(1) \text{ and } f(2) \geq f(3)\}.$$

The *order polynomial* of P is

$$\Omega_P(n) = |\mathcal{O}_P(n)|.$$

Theorem (Stanley)

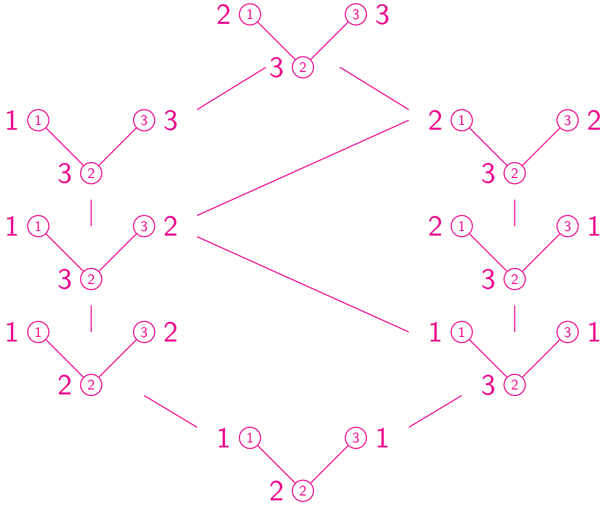
For any P on $[p]$ we have that $\Omega_P(n)$ is a polynomial in n. □

Turn $\mathcal{O}_P(n)$ into a poset by letting

$$f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in P.$$

Ex. With P as in the previous example we have

$\mathcal{O}_P(3) =$



Theorem (Brenti)

For any P on $[p]$, the sequence $(\Omega_P(n))_{n\geq 1}$ is log concave.

Proof (Liang-S). We have that $\mathcal{O}_P(n)$ is a lattice with

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ and } (f \vee g)(x) = \max\{f(x), g(x)\}.$$

Also $\mathcal{O}_P(n)$ is distributive since min distributes over max. Let $L = \mathcal{O}_P(n+1)$ so $|L| = \Omega_P(n+1)$. Define the lower order ideal

$$I = \{f \in L \mid f(x) \leq n \text{ for all } x \in P\}$$

and upper ideal

$$J = \{f \in L \mid f(x) \geq 2 \text{ for all } x \in P\}.$$

Therefore $I \cong J \cong \mathcal{O}_P(n)$ so that $|I| = |J| = \Omega_P(n)$. Similarly $|I \cap J| = \Omega_P(n-1)$. By the Order Ideal Lemma we are done. □

Call $a_0, a_1, a_2, \ldots$ *log-concave at n* if

$$a_n^2 \geq a_{n-1}a_{n+1}.$$

Similarly define being *log-convex at n*. A *generalized Lucas sequence* $(l_n)_{n\geq 0}$ has initial values l_0, l_1 and

$$l_n = l_{n-1} + l_{n-2} \text{ for } n \geq 2.$$

Ex. The Fibonacci sequence $F_0 = F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Call a generalized Lucas sequence $(l_n)_{n\geq 0}$ *positive* if

$$l_0, l_1 > 0.$$

Theorem (Liang-S)

Any positive Lucas sequence has an index shift which is log-concave at odd indices and log-convex at even ones. □

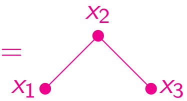
Outline of proof for F_n. For any poset P, let

$$\mathcal{J}(P) = \{I \mid I \text{ is a lower order ideal of } P\}$$

ordered by inclusion.

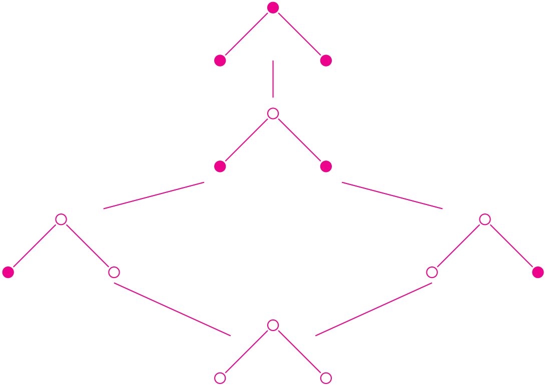
Ex. Suppose

$Z_3 =$



If $\bullet$ and $\circ$ denote elements in and not in an ideal, respectively

$\mathcal{J}(Z_3) =$



Theorem (Fundamental Thm of Finite Distributive Lattices)

For any finite poset P: $\mathcal{J}(P)$ is a distributive lattice. For any finite distributive lattice L there is a poset P with $L \cong \mathcal{J}(P)$. □

The *zigzag poset*, Z_n has elements $x_1, \ldots, x_n$ and relations

$$x_1 < x_2 > x_3 < x_4 > x_5 < \ldots \ .$$

Claim: for $n \geq 0$ we have

$$\#\mathcal{J}(Z_n) = F_{n+1}.$$

To see this, induct on n. Suppose n is even so Z_n ends $x_{n-1} < x_n$. Thus

$$\begin{aligned}\#\mathcal{J}(Z_n) &= \#\{I \in \mathcal{J}(Z_n) \mid x_n \notin I\} \ + \ \#\{I \in \mathcal{J}(Z_n) \mid x_n \in I\} \\ &= \#\{I \in \mathcal{J}(n) \mid I \subseteq Z_{n-1}\} \ + \ \#\{I \in \mathcal{J}(Z_n) \mid x_n, x_{n-1} \in I\} \\ &= \#\mathcal{J}(Z_{n-1}) + \#\mathcal{J}(Z_{n-2}).\end{aligned}$$

To finish proving that F_n alternates log-concavity and -convexity, one now finds appropraite ideals in $\mathcal{J}(Z_{n+1})$ and applies OIL. □

1. Peak polynomials. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$ and $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$. The *peak set* of π is

$$\operatorname{Pk}\pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Ex. If $\pi = 352164 = \pi_1 < \pi_2 > \pi_3 > \pi_4 < \pi_5 > \pi_6$ then

$$\operatorname{Pk}\pi = \{2, 5\}.$$

For any index set I of positive integers, let

$$P_I(n) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Pk}\pi = I\}.$$

Ex. $P_{\{2\}}(4) = \{1324, 1423, 1432, 2314, 2413, 2431, 3412, 3421\}$.

Theorem (Billey-Burdzy-S)

For any I, there is a polynomial $p_I(n)$ such that for $n \geq 0$

$$|P_I(n)| = p_I(n)\ 2^{n-|I|-1}.$$ □

Theorem (Liang-S)

For any I, the sequence $(p_I(n))_{n\geq 0}$ is log-concave. □

2. Stirling numbers. The *Stirling numbers of the second kind* are

$$S(n,k) = \text{number of partitions of } [n] \text{ into } k \text{ subsets (blocks).}$$

By defining a new poset on such partitions which is a distributive lattice we have proved the following.

Theorem (Neuman)

For fixed k the sequence $(S(n,k))_{n\geq 0}$ is log-concave. □

The *(signless) Stirling numbers of the first kind* are

$$c(n,k) = \text{number of permutations of } [n] \text{ having } k \text{ cycles.}$$

Computations support the following conjecture.

Conjecture

Given k, there is an integer N_k such that $(c(n,k))_{n\geq 0}$ is log-concave for $n < N_k$ and log-convex for $n \geq N_k$. □

THANKS FOR LISTENING!