Distributive lattices for proving log-concavity and -convexity

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan
joint work with
Jinting Liang

AMS Sectional Meeting, Milwaukee

Consider a sequence of real numbers

$$(a_n) = (a_n)_{n \geq 0} = a_0, a_1, a_2, \ldots$$

Say that (a_n) is *log-concave* if, for all $n \geq 1$, we have

$$a_n^2 \geq a_{n-1} a_{n+1}.$$

Say that (a_n) is *log-convex* if, for all $n \geq 1$, we have

$$a_n^2 \leq a_{n-1} a_{n+1}.$$

Log-concavity and -convexity have been intensely studied in combinatorics.

A *lower order ideal* of poset $P = (P, \preceq)$ is $I \subseteq P$ such that

$$x \in I \text{ and } y \preceq x \text{ implies } y \in I.$$

An *upper order ideal* of P is $J \subseteq P$ such that

$$x \in J \text{ and } y \succeq x \text{ implies } y \in J.$$

Poset L is a *distributive lattice* if evey $x, y \in L$ have a meet or greatest lower bound, $x \wedge y$, and a join of least upper bound, $x \vee y$, satisfying one of the two equivalent distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in L,$$

or

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \in L.$$

The following follows easily from the FKG (Fortuin, Kasteleyn, Ginibre) inequality.

Lemma (The Order Ideal Lemma, Liang-S)

Let L be a distributive lattice and $I, J \subseteq L$ be ideals.

(a) *If I, J are both lower ideals or both upper ideals then*

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|$$

where $|\ |$ denotes cardinality.

(b) *If one of I, J is a lower order ideal and the other is upper then*

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|.$$

The *Catalan numbers* are

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

So

$$(C_n) = 1, 1, 2, 5, 14, 42, \ldots$$

A *Dyck path of semilength n* is a lattice path P satisfying:

1. P starts at $(0,0)$ and ends at $(2n,0)$,
2. P uses up steps U parallel to $[1,1]$ and down steps D parallel to $[1,-1]$,
3. never goes below the x-axis.

Let

$$\mathcal{D}_n = \{P \mid P \text{ is a Dyck path of semilength } n\}.$$

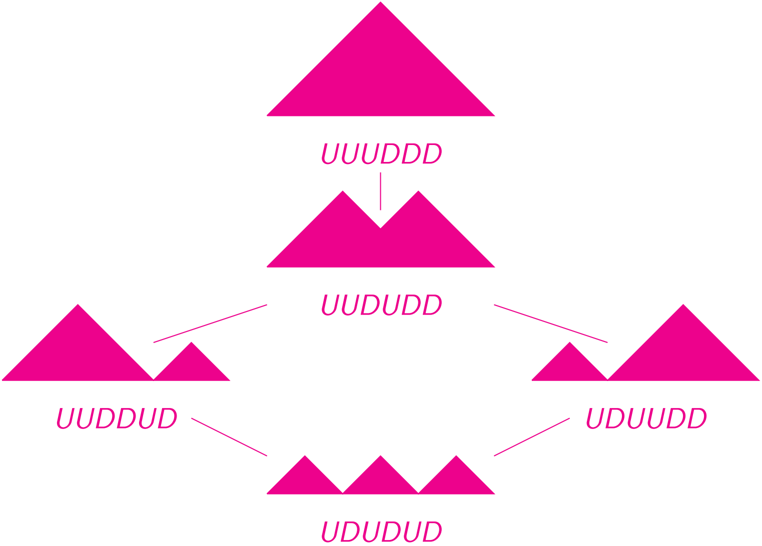
It is well known that

$$C_n = |\mathcal{D}_n|.$$

If $P \in \mathcal{D}_n$ then let $A(P)$ be the physical area enclosed by P and the x-axis. Turn $\mathcal{D}_n$ into a poset by letting

$$P \preceq Q \iff A(P) \subseteq A(Q).$$

Ex. $\mathcal{D}_3$



Lemma (Ferrari, Pinzani)

For all $n \geq 0$ we have that $\mathcal{D}_n$ is a distributive lattice. □

Theorem

The sequence (C_n) is log-convex.

Proof (Liang-S). Let $L = \mathcal{D}_{n+1}$ so that $|L| = C_{n+1}$. The maximal element in $\mathcal{D}_n$ is the path $U^n D^n$. Construct the lower order ideals

$$I = \{P \in L \mid P \preceq U^n D^n UD\}$$

and

$$J = \{P \in L \mid P \preceq UDU^n D^n\}.$$

Therefore $I \cong J \cong \mathcal{D}_n$ so that $|I| = |J| = C_n$. Also

$$I \cap J = \{P \in L \mid P \preceq UDU^{n-1} D^{n-1} UD\}.$$

Therefore $I \cap J \cong \mathcal{D}_{n-1}$ so that $|I \cap J| = C_{n-1}$. Thus, by the Order Ideal Lemma

$$C_n^2 = |I| \cdot |J| \leq |I \cap J| \cdot |L| = C_{n-1} C_{n+1}$$

as desired. □

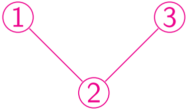
For a positive integer p we let $[p] = \{1, 2, \ldots, p\}$. Let $(P, \preceq)$ be a poset on $[p]$. A *P-partition with range $[n]$* is a map $f : P \to [n]$ such that for all $x \prec y$:

1. $f(x) \geq f(y)$, i.e., f is order reversing, and
2. If $x > y$ then $f(x) > f(y)$.

Let

$$\mathcal{O}_P(n) = \{f \mid \ f \text{ is a } P\text{-partition with range } [n]\}.$$

Ex. If $p = 3$ and

$P =$ 

$$\therefore \mathcal{O}_P(n) = \{f : P \to [n] \mid f(2) > f(1) \text{ and } f(2) \geq f(3)\}.$$

The *order polynomial* of P is

$$\Omega_P(n) = |\mathcal{O}_P(n)|.$$

Theorem (Stanley)

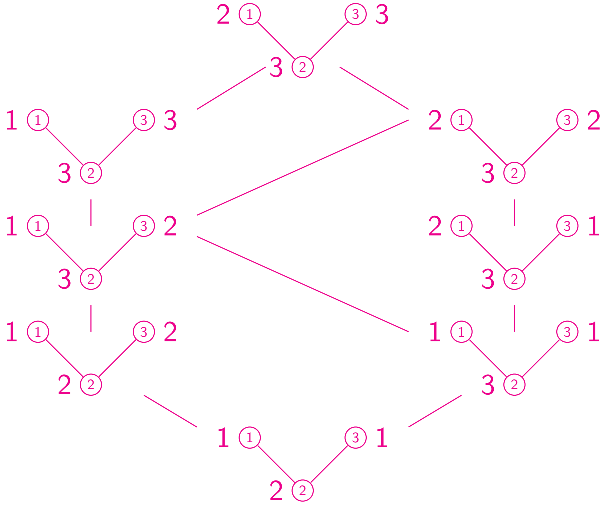
For any P on $[p]$ we have that $\Omega_P(n)$ is a polynomial in n. □

Turn $\mathcal{O}_P(n)$ into a poset by letting

$$f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in P.$$

Ex. With P as in the previous example we have

$\mathcal{O}_P(3) =$



The following result was proved in the special case of naturally labeled P by Chan, Pak, and Panova.

Theorem (Liang-S)

For any P on $[p]$, the sequence $(\Omega_P(n))_{n\geq 1}$ is log concave.

Proof.

We have that $\mathcal{O}_P(n)$ is a lattice with

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ and } (f \vee g)(x) = \max\{f(x), g(x)\}.$$

Also $\mathcal{O}_P(n)$ is distributive since min distributes over max. Let $L = \mathcal{O}_P(n+1)$ so $|L| = \Omega_P(n+1)$. Define the lower order ideal

$$I = \{f \in L \mid f(x) \leq n \text{ for all } x \in P\}$$

and upper ideal

$$J = \{f \in L \mid f(x) \geq 2 \text{ for all } x \in P\}.$$

Therefore $I \cong J \cong \mathcal{O}_P(n)$ so that $|I| = |J| = \Omega_P(n)$. Similarly $|I \cap J| = \Omega_P(n-1)$. By the Order Ideal Lemma we are done. □

1. Fibonacci numbers. Call $a_0, a_1, a_2, \ldots$ *log-concave at n* if

$$a_n^2 \geq a_{n-1}a_{n+1}.$$

Similarly define being *log-convex at n*. The *Fibonacci number* are defined by $F_1 = F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 3$. Using the Fundamental Theorem of Finite Distributive Lattices we prove the following.

Theorem

The sequence (F_n) is log-concave at odd indices and log-convex at even ones. □

2. Stirling numbers. The *Stirling numbers of the second kind* are

$$S(n, k) = \text{number of partitions of } [n] \text{ into } k \text{ subsets (blocks).}$$

By defining a new poset on such partitions which is a distributive lattice we have proved the following.

Theorem (Neuman)

For fixed k the sequence $(S(n,k))_{n\geq 0}$ is log concave. □

The *(signless) Stirling numbers of the first kind* are

$$c(n, k) = \text{number of permutations of } [n] \text{ having } k \text{ cycles.}$$

Computations support the following conjecture.

Conjecture

Given k, there is an integer N_k such that $(c(n,k))_{n\geq 0}$ is log-concave for $n < N_k$ and log-convex for $n \geq N_k$. □

THANKS FOR LISTENING!