Difference d ascent sequences

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February 26, 2024

Ascent sequences and variations

Matrices

Patterns

Posets

Open questions

For $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. A sequence $\alpha = a_1 a_2 \ldots a_n$ of nonnegative integers has *kth prefix*

$$\alpha_k = a_1 a_2 \ldots a_k.$$

Using $\#$ for cardinality, the *ascent set* and *ascent number* of α are

$$\operatorname{Asc} \alpha = \{k \in [n-1] \mid a_{k+1} > a_k\},$$
$$\operatorname{asc} \alpha = \# \operatorname{Asc} \alpha.$$

Call α an *ascent sequence* if

(a1) $a_1 = 0$, and

(a2) $a_{k+1} \leq 1 + \operatorname{asc} \alpha_k$ for $k \in [n-1]$.

For $n \geq 0$, let

$$\mathrm{A}_n = \{\alpha = a_1 a_2 \ldots a_n \mid \alpha \text{ is an ascent sequence}\}.$$

Ex. If $\alpha = 0, 1, 2, 0, 2, 1, 2, 4$ then $\operatorname{Asc} \alpha = \{1, 2, 4, 6, 7\}$, $\operatorname{asc} \alpha = 5$. This α is an ascent sequence, e.g., $a_5 = 2 \leq 1 + \operatorname{asc}(0, 1, 2, 0) = 3$. We have

$$\mathrm{A}_3 = \{000, 001, 010, 011, 012\}.$$

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in 2010. They have since been studied by many authors including Andrews, Baxter, Bényi, Callan, Chen, A. Conway, M. Conway, Dai, Dokos, Duncan, Dwyer, Elvey Price, Fan, Guttmann, Jelínek, Jin, Kubitzke, Lin, Linusson, Mansour, McNamara, Parviainen, Pudwell, Remmel, Sagan, Schlosser, Shattuck, Steingrímsson, Yan, Zhao.

Ascent sequences are in bijection with

1. upper-triangular matrices over $\mathbb{N}$ with nonzero rows and columns.
2. permutations avoiding a bivincular pattern of length 3,
3. unlabeled $(2+2)$-free posets, and
4. matchings with no neighbor nestings.

Sequence $\alpha = a_1 \ldots, a_n$ has *weak ascent set* and *number*

$$\operatorname{wAsc}\alpha = \{k \in [n-1] \mid a_{k+1} \geq a_k\},$$
$$\operatorname{wasc}\alpha = \#\operatorname{wAsc}\alpha.$$

Call α a *weak ascent sequence* if

(w1) $a_1 = 0$, and

(w2) $a_{k+1} \leq 1 + \operatorname{wasc}\alpha_k$ for $k \in [n-1]$.

For $n \geq 0$, let

$$\mathrm{wA}_n = \{\alpha = a_1 a_2 \ldots a_n \mid \alpha \text{ is a weak ascent sequence}\}.$$

Ex. $\alpha = 0, 0, 2, 1, 3 \in \mathrm{wA}_5$, e.g., $a_5 = 3 \leq 1 + \operatorname{wasc}(0, 0, 2, 1) = 3.$

Weak ascent sequences were introduced by Bényi, Claesson and Dukes in 2023. They are in bijection with

1. certain upper-triangular $0, 1$-matrices with nonzero columns.
2. permutations avoiding a bivincular pattern of length 4,
3. certain labeled $(2 + 2)$- and special $(3 + 1)$-free posets, and
4. certain matchings with no left neighbor nestings.

Fix $d \in \mathbb{N}$. Sequence $\alpha = a_1 \ldots, a_n$ has *difference d ascent set* or simply *d-ascent set* and *d-ascent number*

$$\mathrm{dAsc}\,\alpha = \{k \in [n-1] \mid a_{k+1} > a_k - d\},$$
$$\mathrm{dasc}\,\alpha = \#\,\mathrm{dAsc}\,\alpha.$$

Note that when $d = 0$ or 1 we recover ascents or weak ascents, respectively. Call α a *d-ascent sequence* if

(d1) $a_1 = 0$, and

(d2) $a_{k+1} \leq 1 + \mathrm{dasc}\,\alpha_k$ for $k \in [n-1]$.

For $n \geq 0$, let

$$\mathrm{dA}_n = \{\alpha = a_1 a_2 \ldots a_n \mid \alpha \text{ is a } d\text{-ascent sequence}\}.$$

Ex. For $d = 2$ we have $\alpha = 0, 1, 0, 3, 1 \in \mathrm{dA}_4$, e.g., $a_4 = 3 \leq 1 + \mathrm{dasc}(0, 1, 0) = 3$.

We have analogues of all of the bijections for (weak) ascent sequences, but in some cases they become injections. Often our maps are described differently than the previous ones so as to facilitate the generalization.

Given $\alpha = a_1 a_2 \ldots a_n \in \mathrm{dA}_n$, its *d-ascent factorization* is the concatenation

$$\alpha = \delta_0 \delta_1 \ldots \delta_m,$$

where the factors are obtained by dividing α after each d-ascent.
Ex. If $d = 2$ then $\alpha = 0, 1, 0, 3, 0, 4, 2, 0$ has factorization

$$\alpha = 0 \mid 1 \mid 0 \mid 3, 0 \mid 4, 2, 0$$

so

$$\delta_0 = 0,\ \delta_1 = 1,\ \delta_2 = 0,\ \delta_3 = 3, 0,\ \text{and } \delta_4 = 4, 2, 0.$$

Rows and columns of matrices will be indexed by $\{0, 1, \ldots, m\}$. Let Z_m be the zero matrix of such dimensions, and $E_{i,j}$ be the matrix obtained from Z_m by setting the element in row i and column j equal to one. Given α we construct a matrix $\mathrm{mx}(\alpha)$ by constructing a sequence of matrices

$$Z_m = M_0, M_1, \ldots, M_n = \mathrm{mx}(\alpha)$$

where

$$M_k = M_{k-1} + E_{a_k, j} \tag{1}$$

if a_k is in the factor δ_j.

$$M_k = M_{k-1} + E_{a_k,j} \text{ if } a_k \in \delta_j.$$

Ex. Suppose $d = 2$ and $\alpha = 0 \mid 1 \mid 0 \mid 3,1$. Then

$$\delta_0 = 0,\ \ \delta_1 = 1,\ \ \delta_2 = 0,\ \text{ and } \delta_3 = 3,1.$$

So

$$\underset{M_0}{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}} + E_{0,0} \quad \underset{M_1}{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}} + E_{1,1} \quad \underset{M_2}{\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}} + E_{0,2}$$

$$\underset{M_3}{\begin{bmatrix} 1&0&1&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}} + E_{3,3} \quad \underset{M_4}{\begin{bmatrix} 1&0&1&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}} + E_{1,3} \quad \underset{M_5 = \text{mx}(\alpha).}{\begin{bmatrix} 1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&1 \end{bmatrix}}$$

To describe the image of the mx map, if c_j is the jth column of a matrix and is nonzero then let

$$\operatorname{rmin} c_j = \text{smallest row index where } c_j \text{ is nonzero,}$$
$$\operatorname{rmax} c_j = \text{largest row index where } c_j \text{ is nonzero.}$$

For $d \geq 1$ let dMtx_n be the set of all matrices satisfying

(M1) M is upper triangular with entries 0 and 1 and n ones.

(M2) Between any two ones in the same column there are at least $d - 1$ zeros.

(M3) There are no zero columns and for all $j \in [m]$ we have

$$\operatorname{rmax} c_j > \operatorname{rmin} c_{j-1} - d.$$

Note that when $d = 1$, condition (M2) is vacuous.

Theorem (Dukes, S)

For $d \geq 1$, the map $\mathrm{mx} : \mathrm{dA}_n \to \mathrm{dMtx}_n$ is a well-defined bijection.

Let $\mathfrak{S}_n$ be the symmetric group of all permutations $\pi = \pi_1 \dots \pi_n$ of $[n]$. Say π contains permutation $\sigma \in \mathfrak{S}_k$ as a *classical pattern* if there is a subsequence $\kappa = \pi_{i_1} \dots \pi_{i_k}$ of π, called a *copy*, whose elements are in the same relative order as those of σ. In a *bivincular pattern* certain pairs of elements of κ are specified to be in adjacent positions (indicated by a vertical bar between the two in σ), or have adjacent values (indicated by a horizontal bar above the smaller of the two in σ).

Ex. Copies of $\sigma = 231$ in $\pi = 643512$ are 451, 452, 351, and 352. Only one of these is a copy of $2|3\overline{1}$ namely 352.

If σ is any bivincular pattern then its *avoidance set* is

$$\mathrm{Av}_n(\sigma) = \{\pi \in \mathfrak{S}_n \mid \pi \text{ contains no copy of } \sigma\}.$$

Theorem

There are bijections

1. *(Bousquet-Mélou, Claesson, Dukes, Kitaev)* $\mathrm{A}_n \to \mathrm{Av}_n(2|3\overline{1})$, *and*
2. *(Bényi, Claesson, Dukes)* $\mathrm{wA}_n \to \mathrm{Av}_n(3|41\overline{2})$.

Consider the bivincular pattern

$$\sigma_d = (d-1)|d12\ldots\overline{(d-2)}.$$

Theorem (Dukes, S)

For all $d, n \geq 0$ there is an injection $\mathrm{pe} : \mathrm{dA}_n \hookrightarrow \mathrm{Av}_n(\sigma_{d+3})$.

Proof sketch. If $\pi \in \mathrm{Av}_n(\sigma)$ then an *active sites* of π is a place where $n+1$ could be inserted so that the resulting $\pi' \in \mathrm{Av}_{n+1}(\sigma)$. We number the active sites $0, 1, 2, \ldots$ left to right.

Ex. If $\pi = 631245$ and $\sigma = \sigma_4 = 3|41\overline{2}$ then the numbering is

$$\underset{0}{\uparrow}6\ 3\ 1\underset{1}{\uparrow}2\underset{2}{\uparrow}4\underset{3}{\uparrow}5\underset{4}{\uparrow}.$$

Given $\alpha = a_1 \ldots a_n \in \mathrm{dA}_n$, construct a sequence of permutations

$$\emptyset = \pi_0, \pi_1, \ldots, \pi_n = \mathrm{pe}(\alpha)$$

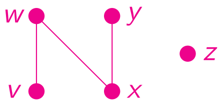
where π_k is obtained from π_{k-1} by inserting k in the active site labeled a_k for $1 \leq k \leq n$.

Ex. If $d = 1$ and $\alpha = 01022$ then

$$\emptyset \quad \underset{0}{\uparrow}1\underset{1}{\uparrow} \quad \underset{0}{\uparrow}1\underset{1}{\uparrow}2\underset{2}{\uparrow} \quad \underset{0}{\uparrow}31\underset{1}{\uparrow}2\underset{2}{\uparrow} \quad \underset{0}{\uparrow}31\underset{1}{\uparrow}2\underset{2}{\uparrow}4\underset{3}{\uparrow} \quad 31254 = \mathrm{pe}(\alpha).$$

Let $(P, \leq_P)$ be a poset. Call P *$(a+b)$-free* if P has no induced subposet isomorphic to the disjoint union of an a-element chain and a b-element chain.

Ex. The following poset is $(2+2)$-free

$P =$ 

$vw \uplus xy$ is a copy of $2+2$ but it is not induced since $x <_P w$.

Theorem (Bousquet-Mélou, Claesson, Dukes, Kitaev)

There is a bijection between A_n and the set of unlabeled, n-element posets which are $(2+2)$-free.

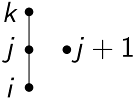
A *factorial poset* is a poset P on $[n]$ satisfying for all $i, j, k \in [n]$

$$i < j \text{ and } j <_P k \implies i <_P k.$$

Theorem (Claesson, Linusson)

1. *The number of factorial posets on $[n]$ is $n!$.*
2. *If P is factorial then P is $(2+2)$-free.*

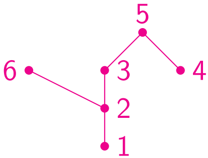
A *special* $3+1$ *poset* is a labeling of $3+1$ of the form



$i < j < k.$

Ex. The poset at right is factorial.
It's not $(3+1)$-free because of $123 \uplus 4$.
It is special $(3+1)$-free.

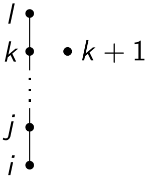
$P =$



Theorem (Bényi, Claesson, Dukes)

There is a bijection between wA_n and the set of factorial posets on $[n]$ which are special $(3+1)$-free.

A *special P_d poset* is a labeling of $(d-1)+1$ of the form

$P_d =$ 

$i < j < \ldots < k < l.$

Theorem (Dukes, S)

For $d \geq 1$ there is an injection from dA_n to the set of factorial posets on $[n]$ which are special P_{d+3}-free.

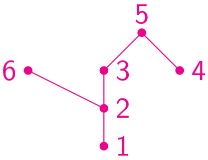
Proof sketch. If $\alpha \in \mathrm{dA}_n$ then define $P = P(\alpha)$ by

$$i <_P j \text{ if and only if } \operatorname{dasc}\alpha_i < a_j.$$

Ex. If $d = 2$ and $\alpha = 012032$ then

i	1	2	3	4	5	6
a_i	0	1	2	0	3	2
$\operatorname{dasc}\alpha_i$	0	1	2	2	3	4

If $j = 6$ then $a_6 = 2$. And $\operatorname{dasc}\alpha_1, \operatorname{dasc}\alpha_2 < 2$ so $1, 2 <_P 6$.

$P(\alpha) =$ 

Revisiting ascent sequences. When $d = 0$, so α is an ascent sequence, one can define a matrix $\mathrm{mx}(\alpha)$ in exactly the same way as for $d \geq 1$. But now the same $E_{i,j}$ can be added many times. For $d = 0$ let Mtx_n be the set of all matrices M with

(M1) M is upper triangular with entries in $\mathbb{N}$ summing to n.

(M3) There are no zero columns and for all $j \in [m]$ we have $\mathrm{rmax}\, c_j > \mathrm{rmin}\, c_{j-1} - d$.

Theorem (Dukes, S)

The map $\mathrm{mx} : \mathrm{A}_n \to \mathrm{Mtx}_n$ *is a well-defined bijection.*

The set Mtx'_n of *Fishburn matrices* are those satisfying (M1) and

(M3') There are no zero rows or columns.

Theorem (Dukes, Parviainen)

There is a bijection $\mathrm{mx}' : \mathrm{A}_n \to \mathrm{Mtx}'_n$.

Question

1. *Is there a simple, direct bijection* $\mathrm{Mtx}_n \to \mathrm{Mtx}'_n$*?*
2. *Is there a subset of the factorial posets defined by avoidance in bijection with* A_n*?*

From injections to bijections. If Σ is a set of patterns then let

$$\mathsf{Av}_n(\Sigma) = \{\pi \in \mathfrak{S}_n \mid \pi \text{ avoids every } \sigma \in \Sigma\}.$$

Question

1. *Is there a bijection between* dA_n *and* $\mathsf{Av}_n(\Sigma)$ *for some* Σ *containing* σ_{d+3}*?*
2. *What about the analogous question for posets?*

Series. The following result was derived using the kernel method.

Theorem (Bousquet-Mélou, Claesson, Dukes, Kitaev)

$$\sum_{n\geq 0} \#\,\mathrm{A}_n\, t^n = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-t)^i\right).$$

This series had been shown earlier by Zagier to count matchings without nestings.

Question

Find the generating function $\sum_{n\geq 0} \#\,\mathrm{dA}_n\, t^n$*.*

THANK YOU FOR
HELPING CELEBRATE
MY BIRTHDAY/RETIREMENT!