The cyclic sieving phenomenon - an introduction

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Outline

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2. Recent work by: Bessis, Eu, Fu, Petersen, Pylyavskyy, Rhoades, Sagan, Serrano, Shareshian, Wachs.

Running example. Let $[n] = \{1, 2, \ldots, n\}$ and consider the set of multisets

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The action of $(1, 2, 3)$ is

$$\begin{array}{lll} (1,2,3)11 = 22 & (1,2,3)22 = 33 & (1,2,3)33 = 11 \\ (1,2,3)12 = 23 & (1,2,3)13 = 21 = 12 & (1,2,3)23 = 31 = 13. \end{array}$$

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If $m \equiv n \pmod{d}$ *and* $\omega = \omega_d$, *then*

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Outline

If G acts on $S = \{s_1, \ldots, s_k\}$ then G also acts on the vector space

$$V = \mathbb{C}S = \{c_1\mathbf{s}_1 + \cdots + c_k\mathbf{s}_k \ : \ c_i \in \mathbb{C} \text{ for all } i\}.$$

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And so

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Outline

Consider the space of k-fold *symmetric tensors* over $V = \mathbb{C}[n]$:

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So $\chi'(g) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 = h_2(x_1, x_2, x_3)$

Lemma

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Proof Induct on n. ■

The linear map $[(1, 2, \ldots, n)]$ has characteristic polynomial $x^n - 1$ with roots $1, \omega_n, \ldots, \omega_n^{n-1}$. So $\exists$ a basis B of $\mathbb{C}[n]$ with

$$[(1, 2, \ldots, n)]_B = \mathrm{diag}(1, \omega_n, \ldots, \omega_n^{n-1}).$$

Lemma

$h_k(1, q, q^2, \ldots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q .$

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Also $\mathrm{Sym}_k(n) \cong \mathbb{C}(\binom{[n]}{k})$, so for all $g \in G$

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Comparing (3) and (4) gives the c.s.p. ∎