Graph Coloring and Symmetric Functions in Noncommuting Variables

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0. History

1. Chromatic Polynomials

2. Stanley's Symmetric Function

3. Noncommuting Variables

4. Basis Change à la Doubilet

5. MacMahon and Schur Symmetric Functions

0. History

1932	H. Whitney
1936	M. C. Wolf
1969	G M. Bergman & P. M. Cohn
1981	M.-P. Schützenberger & A. Lascoux
1994 & 1995	R. P. Stanley
1995	I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, & J.-I. Thibon
1998	S. Fomin & C. Greene
2000 & 2001	D. Gebhard & BES
2002	M. Rosas & BES

1. Chromatic Polynomials

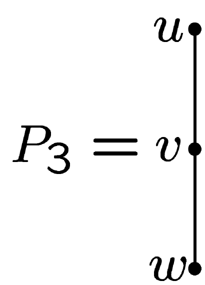
Let $G = (V, E)$ be a graph with vertices V and edges E. Loops and multiedges are permitted. A *proper coloring of* G is $c : V \to \{1, 2, \ldots, n\}$ such that

$$uv \in E \Rightarrow c(u) \neq c(v).$$

The *chromatic polynomial of* G is

$$P(G) = P(G, n) := \#\text{ of proper } c : V \to \{1, 2, \ldots, n\}.$$

Ex. The path P_3.



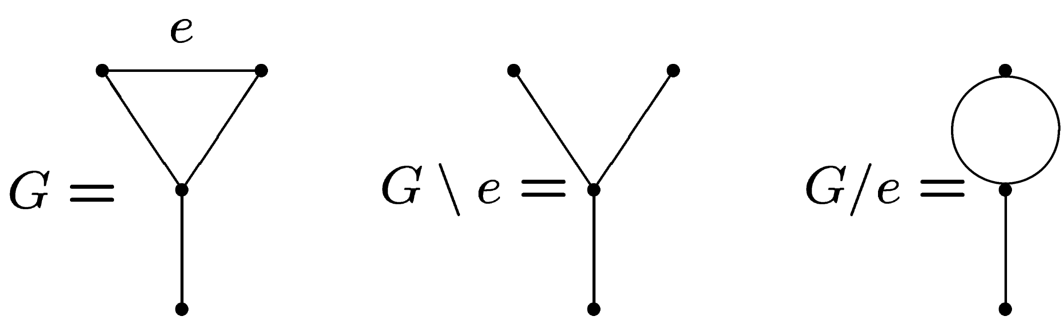
$$\begin{aligned} P(G, n) &= (\#\text{ of ways to color } u) \\ &\quad \cdot(\#\text{ of ways to then color } v) \\ &\quad \cdot(\#\text{ of ways to then color } w) \\ &= n(n-1)(n-1) \\ &= n^3 - 2n^2 + n. \end{aligned}$$

1a. Deletion/Contraction

If $e \in E$ then let

$$\begin{aligned} G \setminus e &= G \text{ with } e \text{ deleted,} \\ G/e &= G \text{ with } e \text{ contracted to a point.} \end{aligned}$$

Ex. Deletion and contraction in a specific graph.



Lemma 1 (Deletion-Contraction) *If $e \in E$ then*

$$P(G) = P(G \setminus e) - P(G/e).$$

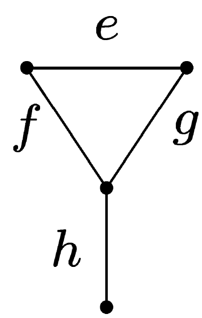
Proof If $e = uv$ then

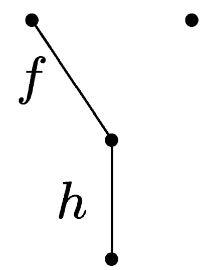
$$\begin{aligned} P(G \setminus e) &= (\#\text{ of colorings with } c(u) \neq c(v)) \\ &\quad +(\#\text{ of colorings with } c(u) = c(v)) \\ &= P(G) + P(G/e). \qquad \blacksquare \end{aligned}$$

1b. Whitney's Theorem

If $F \subseteq E$ then let F also stand for the subgraph of G with vertices V and edges F.

Ex. The two meanings of F.

$G =$ 

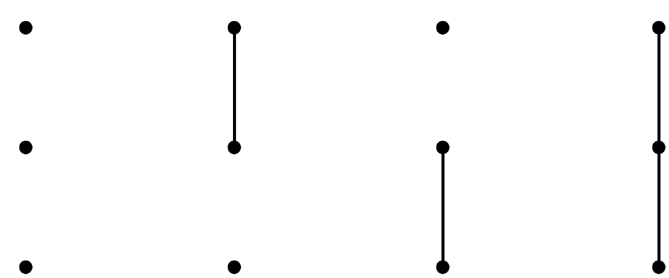
$F = \{f, h\} =$ 

Theorem 2 (Whitney, 1932) *We have*

$$P(G) = \sum_{F \subseteq E} (-1)^{|F|}\, n^{k(F)}$$

where $|F|$ *is the cardinality of* F*, and* $k(F)$ *is the number of connected components of* F*.* ■

Ex. The path P_3.

F : 

$$\begin{aligned} \sum(-1)^{|F|}\, n^{k(F)} &= n^3 \quad - \quad n^2 \quad - \quad n^2 \quad + \quad n \\ &= n^3 - 2n^2 + n. \end{aligned}$$

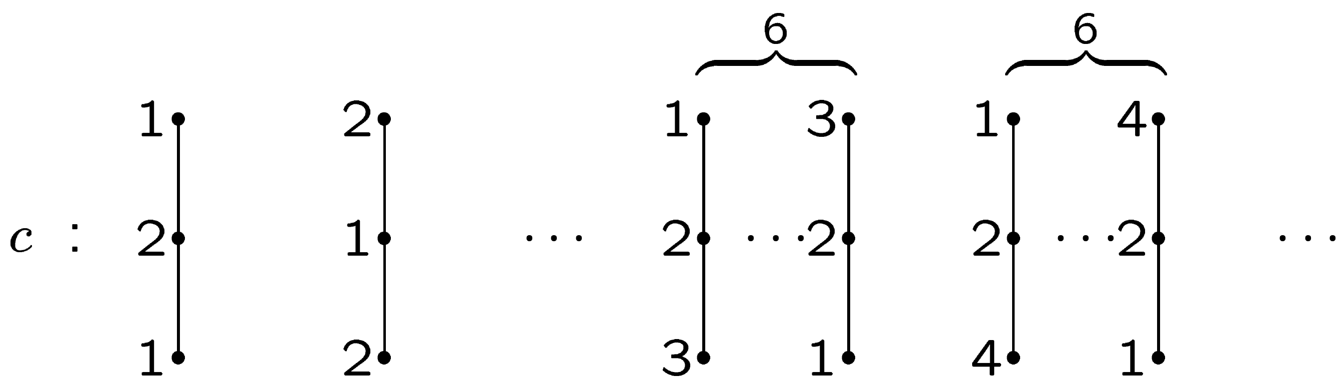
2. Stanley's Symmetric Function

Let $V = \{v_1, \ldots, v_d\}$ and let $x = \{x_1, x_2, \ldots\}$ be a set of commuting variables. Define

$$X(G) = X(G, x) := \sum x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_d)}$$

where the sum is over all proper $c : V \to \{1, 2, \ldots\}$.

Ex. The path P_3.



$$X(G) = x_1^2 x_2 + x_1 x_2^2 + \cdots + 6x_1x_2x_3 + 6x_1x_2x_4 + \cdots$$

Notes.

1. Suppose that we let $x_1 = x_2 = \cdots = x_n = 1$ and $x_i = 0$ for $i > n$, denoted $x = 1^n$, then

$$X(G, 1^n) \;=\; \sum_{\substack{c:V \to \{1,2,\ldots,n\} \\ c \text{ proper}}} 1 \;=\; P(G, n).$$

2. $X(G, x)$ is a symmetric function.

2a. Symmetric Functions

Let S_m be the symmeteric group on $\{1, 2, \ldots, m\}$ and let $\mathbb{Q}[[x]]$ denote the algebra of formal power series in x over $\mathbb{Q}$. Now $g \in S_m$ acts of $f(x) \in \mathbb{Q}[[x]]$ by

$$gf(x_1, x_2, \ldots) = f(x_{g1}, x_{g2}, \ldots)$$

where $gi = i$ for $i > m$. Then f is *symmetric* if

$$gf = f \quad \text{for all } g \in S_m \text{ and all } m \geq 1.$$

An *integer partition of* d, $\lambda \vdash d$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum_i \lambda_i = d$. The associated *monomial symmetric function* is

$$m_\lambda = \text{sum of all monomials with exponent } \lambda.$$

Ex. If $\lambda = (3, 3, 1)$ then

$$m_{(3,3,1)} = x_1^3 x_2^3 x_3 + x_1^3 x_2 x_3^3 + x_1 x_2^3 x_3^3 + x_1^3 x_2^3 x_4 + \cdots$$

The *algebra of symmetric functions* is

$$\Lambda = \Lambda(x) = \text{span}\{m_\lambda \ : \ \text{all } \lambda\}.$$

Another basis for Λ consists of the *power sum symmetric functions*, p_λ, defined as follows:

$$\begin{array}{lcll} p_d & = & x_1^d + x_2^d + x_3^d + \cdots & \text{for } d \text{ a positive integer,} \\ p_\lambda & = & p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k} & \text{for } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k). \end{array}$$

Ex. $$\begin{array}{rcl} p_{(3,3,1)} & = & p_3 \ p_3 \ p_1 \\ & = & (x_1^3 + x_2^3 + \cdots)^2 (x_1 + x_2 + \cdots). \end{array}$$

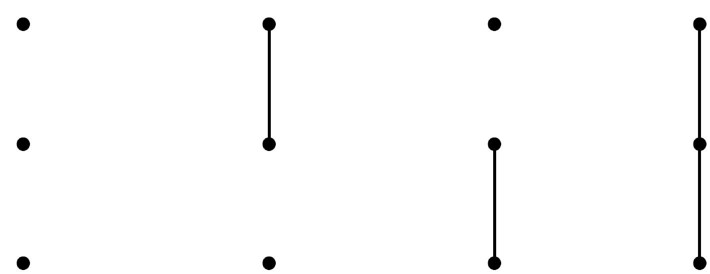
Theorem 3 (Stanley, 1995) *We have*

$$X(G) = \sum_{F \subseteq E} (-1)^{|F|} \, p_{\lambda(F)}$$

where $\lambda(F) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$*,* λ_i *being the number of vertices in the* i*th component of* F*.* ■

Ex. If $G = P_3$ then

F :



$$\sum (-1)^{|F|} \, p_{\lambda(F)} \ = \ p_{(1,1,1)} - p_{(2,1)} - p_{(2,1)} + p_{(3)}$$

Stanley implies Whitney: Substituting $x = 1^n$

$$\begin{array}{rcl} p_d(1^n) & = & \overbrace{1 + 1 + \cdots + 1}^{n} = n \quad \text{and} \\ p_\lambda(1^n) & = & \overbrace{n \cdot n \cdots n}^{k(F)} = n^{k(F)}. \end{array}$$

3. Noncommuting Variables

Let $x_1, x_2, \ldots$ be noncommuting and let $\mathbb{Q}\langle\langle x \rangle\rangle$ be the corresponding ring of formal power series. Fix a total order $v_1, v_2, \ldots, v_d$ on V and define

$$Y(G) = Y(G, x) := \sum x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_d)}$$

where the sum is over all proper $c : V \to \{1, 2, \ldots\}$.

Define an operation of *induction*, $\uparrow$, by

$$x_{i_1} x_{i_2} \cdots x_{i_d} \uparrow = \; x_{i_1} x_{i_2} \cdots x_{i_d} x_{i_d}$$

and linear extension to $\mathbb{Q}\langle\langle x \rangle\rangle$.

Ex. $x_2^3 x_1 x_3^2 \uparrow = \; x_2^3 x_1 x_3^3$.

Lemma 4 (Gebhard-S, 2000) *If* $v_{d-1} v_d \in E$ *then*

$$Y(G) = Y(G \setminus e) - Y(G/e) \uparrow . \quad \blacksquare$$

3a. Symmetric Functions in Noncommuting Variables

The action of S_m on $\mathbb{Q}\langle\langle x\rangle\rangle$ is the same as on $\mathbb{Q}[[x]]$. The symmetric functions in noncommuting variables have bases indexed by partitions of $[d] := \{1, 2, \ldots\}$.

A *set partition of* $[d]$, $\pi \vdash [d]$, is a collection of sets $\pi = B_1/B_2/\ldots/B_k$ such that $\uplus_i B_i = [d]$. The B_i are called the *blocks* of the set partition. The *type of* π is the integer partition

$$\lambda(\pi) = (|B_1|, |B_2|, \ldots, |B_k|).$$

Ex. $\pi = 143/268/57 \vdash [8]$ is of type $\lambda(\pi) = (3, 3, 2)$.

The associated *monomial symmetric function* is

$m_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_d}$ where
$i_j = i_k$ iff j, k are in the same block of π.

Ex. $m_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + \cdots$.

The *algebra of symmetric functions in noncommuting variables* is

$$\Pi = \Pi(x) = \text{span}\{m_\pi \ : \ \text{all } \pi\}.$$

The *power sum symmetric function* basis for $\Pi(x)$ is

$$p_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_d} \text{ where}$$
$$i_j = i_k \text{ if } j, k \text{ are in the same block of } \pi.$$

Ex. $p_{13/2} = m_{13/2} + x_1^3 + x_2^3 + \cdots$.

An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash d$ can be written

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, d^{m_d})$$

where m_i is the multiplicity of i in λ. Define

$$\begin{aligned} \lambda! &= \lambda_1! \lambda_2! \cdots \lambda_k! \\ \lambda^! &= m_1! m_2! \cdots m_d! \end{aligned}$$

Let $\rho : \mathbb{Q}\langle\langle x \rangle\rangle \to \mathbb{Q}[[x]]$ be the forgetful map.

$$\rho(m_\pi) = \lambda(\pi)^! m_{\lambda(\pi)} \quad \text{and} \quad \rho(p_\pi) = p_{\lambda(\pi)}.$$

Theorem 5 (Gebhard-S, 2000) *If* $V = [d]$ *then*

$$Y(G) = \sum_{F \subseteq E} (-1)^{|F|} \, p_{\pi(F)}$$

where $\pi(F) = B_1/B_2/\ldots/B_k$, B_i *being the vertices in the* i*th component of* F. ■

3b. The (3+1)-Free Conjecture

Let C_n denote a totally ordered set on n elements. A poset (partially ordered set) P is $(n+m)$-*free* if it contains no induced subposet isomorphic to $C_n \uplus C_m$. The *incomparability graph* of a poset P is G_P where $V = P$ with an edge from u to v if u and v are not related in P. Let $\{e_\lambda \ : \ \text{all } \lambda\}$ be the elementary symmetric function basis for Λ.

Conjecture 6 (Stanley and Stembridge, 1993) *If P is (3+1)-free and $X(G_P) = \sum_\lambda c_\lambda e_\lambda$ then $c_\lambda \geq 0$ for all λ. (We say that $X(G_P)$ is e-positive.)*

The *indifference graphs* are the G_P where P is both $(3+1)$-free and $(2+2)$-free. They can also be characterized as follows. Let $\mathcal{I} = (I_1, I_2, \ldots, I_k)$ be a sequence of intervals $I_j \subseteq [d]$. The corresponding indifference graph, $G_\mathcal{I}$, has vertex set $V = [d]$ with a complete graph on the vertices of I_j for each j. WLOG, there is no containment among the I_j and they are listed in order of increasing left endpoint.

Theorem 7 (Gebhard and S, 2000) *If we have $|I_j \cap I_{j+1}| \leq 1$ for all j, then $X(G_\mathcal{I})$ is e-positive.* ■

4. Basis Change à la Doubilet

A basis b_λ for $\Lambda(x)$ is *multiplicative* if it's defined by

1. defining b_d for positive integers d,
2. defining $b_\lambda = b_{\lambda_1} b_{\lambda_2} \cdots b_{\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

The following bases are multiplicative: the *power sum symmetric functions*

$$p_d = x_1^d + x_2^d + \cdots,$$

the *elementary symmetric functions*

$$e_d = \text{sum of all square-free monomials of degree } d,$$

the *complete homogeneous symmetric functions*

$$h_d = \text{sum of all monomials of degree } d,$$

Ex. We have

$$\begin{aligned} p_{(3,3,1)} &= (x_1^3 + x_2^3 + \cdots)^2 (x_1 + x_2 + \cdots) \\ e_{(3,3,1)} &= (x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots)^2 (x_1 + x_2 + \cdots) \\ h_{(3,3,1)} &= (x_1^3 + x_1^2 x_2 + x_1 x_2 x_3 + \cdots)^2 (x_1 + x_2 + \cdots) \end{aligned}$$

Note that the *monomial symmetric fuctions*

$$m_\lambda = \text{sum of all monomials with exponent } \lambda$$

do not form a multiplicative basis.

In $\Pi(x)$ we have corresponding bases

$$p_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_d} \text{ where}$$
$$i_j = i_k \text{ if } j,k \text{ are in the same block of } \pi,$$

and

$$e_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_d} \text{ where}$$
$$i_j \neq i_k \text{ if } j,k \text{ are in the same block of } \pi.$$

Recall also

$$m_\pi = \sum x_{i_1} x_{i_2} \cdots x_{i_d} \text{ where}$$
$$i_j = i_k \text{ iff } j,k \text{ are in the same block of } \pi.$$

Ex. We have

$$\begin{aligned}
m_{13/2} &= x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + \cdots, \\
p_{13/2} &= x_1x_2x_1 + x_2x_1x_2 + \cdots + x_1^3 + x_2^3 + \cdots \\
&= m_{13/2} + m_{123}, \\
e_{13/2} &= x_1x_1x_2 + \cdots + x_1x_2x_2 + \cdots + x_1x_3x_2 + \cdots \\
&= m_{12/3} + m_{1/23} + m_{1/2/3}.
\end{aligned}$$

If $\rho : \mathbb{Q}\langle\langle x \rangle\rangle \to \mathbb{Q}[[x]]$ is the forgetful map.

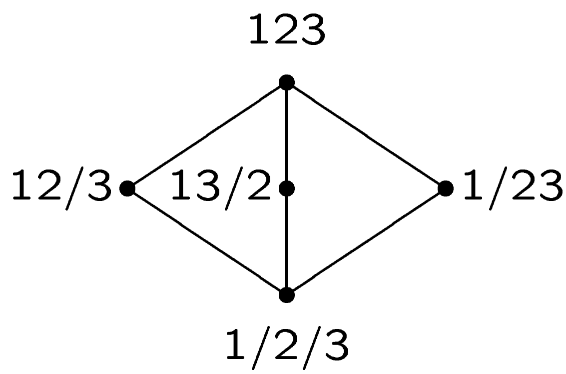
$$\begin{aligned}
\rho(m_\pi) &= \lambda(\pi)^! m_{\lambda(\pi)} \\
\rho(p_\pi) &= p_{\lambda(\pi)} \\
\rho(e_\pi) &= \lambda(\pi)! e_{\lambda(\pi)}.
\end{aligned}$$

4a. The Partition Lattice and Möbius Inversion

Let Π_d be the poset of all $\pi \vdash [d]$ ordered by *refinement*, i.e., if $\pi = B_1/\ldots/B_k$ and $\sigma = C_1/\ldots/C_l$ then $\pi \le \sigma$ if each B_i is contained in some C_j.

Ex. We have

$\Pi_3 =$



Π_d is a *lattice*, meaning every pair $\pi, \sigma \in \Pi_d$ has

$$\begin{aligned} \pi \wedge \sigma &= \textit{greatest lower bound} \text{ or } \textit{meet} \text{ of } \pi \text{ and } \sigma, \\ \pi \vee \sigma &= \textit{least upper bound} \text{ or } \textit{join} \text{ of } \pi \text{ and } \sigma. \end{aligned}$$

So Π_d has a unique minimal element $\hat{0} = 1/2/\ldots/d$. Immediately from the definitions

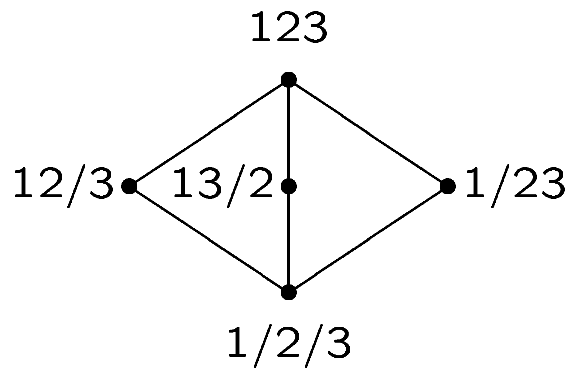
$$p_\pi = \sum_{\sigma \ge \pi} m_\sigma \quad \text{and} \quad e_\pi = \sum_{\sigma \wedge \pi = \hat{0}} m_\sigma .$$

To invert these sums, define the *Möbius function* of a finite poset P to be $\mu : P \times P \to \mathbb{Z}$ such that

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = b, \\ -\displaystyle\sum_{a \le c < b} \mu(a,c) & \text{else.} \end{cases}$$

Ex.

$\mu(\hat{0}, \pi)$:



In general, $\mu(\hat{0}, B_1/\ldots/B_k) = \prod_i (-1)^{|B_i|-1}(|B_i|-1)!$

Theorem 8 (Möbius Inversion, Rota, 1964) *Let P be a finite poset and G be an additive group. If $f, g : P \to G$ then*

$$f(a) = \sum_{b \ge a} g(b) \text{ for all } a \in P$$

$$\text{iff } g(a) = \sum_{b \ge a} \mu(a,b) f(b) \text{ for all } a \in P.$$

Corollary 9 (Doubilet, 1972) *We have*

$$m_\pi = \sum_{\sigma \ge \pi} \mu(\pi, \sigma) p_\sigma, \quad m_\pi = \sum_{\sigma \ge \pi} \frac{\mu(\pi,\sigma)}{\mu(\hat{0},\sigma)} \sum_{\tau \le \sigma} \mu(\tau,\sigma) e_\tau.$$

What about h_π? A function $f : D \to R$ has *kernal* $\ker f \vdash D$ whose blocks are the nonempty sets among the $f^{-1}(r)$ for $r \in R$. Also, $f : [d] \to x$ has associated *monomial*

$$M_f = f(1)f(2)\cdots f(d).$$

Then

$$m_\pi = \sum_{\ker f = \pi} M_f.$$

Ex. If $\pi = 13/2$ then

$$\begin{array}{ccccccc} f & : & \begin{array}{c} f(13)=x_1, \\ f(2)=x_2 \end{array} & & \begin{array}{c} f(13)=x_2, \\ f(2)=x_1 \end{array} & & \cdots \\ \sum M_f & = & x_1x_2x_1 & + & x_2x_1x_2 & + & \cdots \end{array}$$

Define

$$h_\pi = \sum_{(f,L)} M_f$$

where $f : [d] \to x$ and L is a linear ordering of the elements of each block of $\ker f \wedge \pi$. Then

$$\rho(h_\pi) = \lambda(\pi)! h_{\lambda(\pi)}.$$

Ex. If $\pi = 13/2$ then

$$\begin{array}{ccccccc} (f,L) & : & \begin{array}{c} f(13)=x_1, \\ f(2)=x_1 \end{array} & & \begin{array}{c} f(31)=x_1, \\ f(2)=x_1 \end{array} & & \cdots \\ \sum M_f & = & x_1^3 & + & x_1^3 & + & \cdots \end{array}$$

5. MacMahon and Schur Symmetric Functions

Consider the d sets of variables $x' = \{x'_1, x'_2, \ldots\}$, $x'' = \{x''_1, x''_2, \ldots\}$, ..., $x^{(d)} = \{x^{(d)}_1, x^{(d)}_2, \ldots\}$. Let $g \in S_m$ act on $f(x', x'', \ldots, x^{(d)}) \in \mathbb{Q}[[x', x'', \ldots, x^{(d)}]]$ *diagonally*:

$$gf(x'_1, x''_1, \ldots, x'_2, x''_2, \ldots) = f(x'_{g1}, x''_{g1}, \ldots, x'_{g2}, x''_{g2}, \ldots).$$

The monomial

$$x'^{a_1}_1 x''^{b_1}_1 \cdots x^{(d)c_1}_1 x'^{a_2}_2 x''^{b_2}_2 \cdots x^{(d)c_2}_2 \cdots$$

has *multiexponent*

$$\begin{aligned} \vec{\lambda} &= \{\lambda^1, \lambda^2, \ldots\} \\ &:= \{(a_1, b_1, \ldots, c_1), (a_2, b_2, \ldots, c_2), \ldots\} \end{aligned}$$

as well as *multidegree*

$$\begin{aligned} \vec{n} &= (n_1, n_2, \ldots, n_d) \\ &:= (a_1, b_1, \ldots, c_1) + (a_2, b_2, \ldots, c_2) + \cdots \end{aligned}$$

and we write $\vec{\lambda} \vdash \vec{n}$. The associated *monomial MacMahon symmetric function* is

$$m_{\vec{\lambda}} = \text{sum of all monomials with multiexponent } \vec{\lambda}.$$

Ex. $m_{\{(2,1),(3,0)\}} = x'^2_1 x''_1 x'^3_2 + x'^3_1 x'^2_2 x''_2 + \cdots$

The *algebra of MacMahon symmetric functions* is

$$\mathcal{M} = \mathcal{M}(x', x'', \ldots, x^{(d)}) = \text{span}\{m_{\vec{\lambda}} \ : \ \text{all } \vec{\lambda}\}.$$

Consider

$$\mathcal{M}_{(1^d)} = \text{span}\{m_{\vec{\lambda}} \ : \ \text{all } \vec{\lambda} \vdash (\overbrace{1,1,\ldots,1}^{d})\}.$$

Now there exists an isomorphism of vector spaces $\phi : \bigoplus_{d \geq 0} \mathcal{M}_{(1^d)} \to \Pi$ given by

$$x'_i x''_j \cdots x^{(d)}_k \mapsto x_i x_j \cdots x_k$$

and linear extension. Also, under this isomorphism $m_{\vec{\lambda}} \mapsto m_\pi$ where λ_i is the characteristic vector of B_i.

Ex. $m_{\{(1,0,1,1,0),(0,1,0,0,1)\}} \mapsto m_{134/25}$.

Notes 1. This is *not* an isomorphism of algebras as $\bigoplus_{d \geq 0} \mathcal{M}_{(1^d)}$ is commutative.

2. The multiplicative bases of Λ have analogs in $\mathcal{M}$. For example, $h_{\vec{\lambda}} = h_{\{\lambda^1\}} h_{\{\lambda^2\}} \cdots$ where $h_{\{(n_1,n_2,\ldots,n_d)\}}$ has the generating function

$$\sum_{n_1,n_2,\ldots,n_d \geq 0} h_{\{(n_1,n_2,\ldots,n_d)\}} r^{n_1} s^{n_2} \ldots t^{n_d}$$

$$= \prod_{i \geq 1} \frac{1}{1 - x'_i r - x''_i s - \cdots - x^{(d)}_i t}.$$

5a. Schur Functions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ has a *shape*, also denoted λ, consisting of k left-justified rows with λ_i dots in row i.

Ex. Partition $\lambda = (3,3,2,1)$ has Ferrers diagram

$$\lambda = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array} .$$

A *(semistandard) Young tableau T of shape λ*, written $\lambda(T) = \lambda$, is obtained by replacing each dot of the shape of λ with a positive integer so that rows weakly increase and columns strictly increase.

Ex. A Young tableau of shape $(3,3,2,1)$ is

$$T = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 3 & 6 & \\ 5 & & \end{array} .$$

The *Schur function*, s_λ, is

$$s_\lambda = \sum_{\lambda(T)=\lambda} M_T \quad \text{where} \quad M_T = \prod_{i \in T} x_i.$$

Ex. if $\lambda = (2,1)$ then

$$T \quad : \quad \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \; , \; \begin{matrix} 1 & 2 \\ 2 & \end{matrix} \; , \; \ldots \; , \; \begin{matrix} 1 & 2 \\ 3 & \end{matrix} \; , \; \begin{matrix} 1 & 3 \\ 2 & \end{matrix} \; , \; \ldots$$

$$s_{(2,1)} \;=\; x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_3 + \cdots$$

If $\lambda, \vec{n} \vdash d$ then a *primed Young tableau T' of shape λ and multidegree $\vec{n}$* is obtained from T by putting primes on n_1 elements of T, double primes on n_2 elements of T, etc. The corresponding *MacMahon Schur function* is

$$S_\lambda^{\vec{n}} = \sum_{\lambda(T')=\lambda} M_{T'} \quad \text{where} \quad M_{T'} = \prod_{i^{(j)} \in T} x_i^{(j)}.$$

Ex. if $\lambda = (2,1)$ and $\mu = (1,1,1)$ then

$$T \quad : \quad \begin{matrix} 1' & 1'' \\ 2''' & \end{matrix} \; , \; \begin{matrix} 1'' & 1' \\ 2''' & \end{matrix} \; , \; \ldots \; , \; \begin{matrix} 1''' & 1'' \\ 2' & \end{matrix} \; , \; \ldots$$

$$S_{(2,1)}^{(1,1,1)} \;=\; x_1' x_1'' x_2''' + x_1' x_1'' x_2''' + \cdots + x_1'' x_1''' x_2' + \cdots$$

The MacMahon Schur functions have many of the same properties as do the regular Schur functions.

Theorem 10 (Jacobi-Trudi Determinant) *For the partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$*, we have*

$$s_\lambda = |h_{\lambda_i - i + j}|. \quad \blacksquare$$

Define $\vec{m} \leq \vec{n}$ to be the componentwise partial order on vectors of length d.

Theorem 11 (Rosas-S,2001) *Given two partitions* $\lambda, \vec{n} \vdash d$*, we have*

$$S_\lambda^{\vec{n}} = \left| \sum_{\substack{\vec{m} \leq \vec{n} \\ \vec{m} \vdash \lambda_i - i + j}} h_{\vec{m}} \right|$$

with the convention that if the product of two monomials does not have multidegree $\vec{n}$ *then that product is zero.* $\blacksquare$