Cyclic Shuffle Compatibility

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Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. Consider $S \subseteq \mathbb{P}$ with $|S|$, the *cardinality* of S, finite. A *(linear) permutation* of S is a linear ordering $\pi = \pi_1\pi_2 \ldots \pi_n$ of the set S.

Ex. If $S = \{2, 4, 7\}$ then one possible permutation is $\pi = 472$.

A *statistic* is a function st whose domain is all permutations. Examples include the *descent set* of $\pi = \pi_1 \ldots \pi_n$

$$\operatorname{Des} \pi = \{i \ : \ \pi_i > \pi_{i+1}\} \subseteq [n-1],$$

and the *descent number* of π

$$\operatorname{des} \pi = |\operatorname{Des} \pi|.$$

Note that $\operatorname{des} \pi$ in the number of copies of the consecutive pattern 21 in π.

Ex. If $\pi = 73698 = \pi_1 > \pi_2 < \pi_3 < \pi_4 > \pi_5$ then

$$\operatorname{Des} \pi = \{1, 4\}, \quad \operatorname{des} \pi = 2.$$

If π, σ are permutations with $\pi \cap \sigma = \emptyset$ then their *shuffle set* is

$$\pi \sqcup\!\sqcup\, \sigma = \{\tau \ : \ |\tau| = |\pi| + |\sigma| \text{ and } \pi, \sigma \text{ are subwords of } \tau\}.$$

Ex. We have

$$25 \sqcup\!\sqcup\, \mathbf{74} = \{25\mathbf{74},\ 2\mathbf{7}5\mathbf{4},\ 2\mathbf{74}5,\ \mathbf{7}25\mathbf{4},\ \mathbf{7}2\mathbf{4}5,\ \mathbf{74}25\}.$$

Note that

$$\operatorname{des}(25 \sqcup\!\sqcup\, 74) = \{\{1, 1, 1, 2, 2, 2\}\} = \operatorname{des}(12 \sqcup\!\sqcup\, 43).$$

Statistic st is *shuffle-compatible* if the multiset $\operatorname{st}(\pi \sqcup\!\sqcup\, \sigma)$ depends only on $\operatorname{st}\pi, \operatorname{st}\sigma, |\pi|$, and $|\sigma|$.

Theorem (Stanley)

Both Des *and* des *are shuffle-compatible.* □

Shuffle-compatibility is implicit in the work of Stanley on *P*-partitions. It was explicitly defined and studied using algebras whose multiplication involves shuffles by Gessel and Zhuang. Further work in the linear case was done by Grinberg, by Oğuz, and by Baker-Jarvis and S.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$. Monomial $x_i^a x_j^b \cdots x_k^c$ with $i < j < \ldots < k$, has *exponent sequence* $ab \ldots c$, and *degree* $= a + b + \cdots + c$.

Ex. $x_1^2 x_3^4 x_6^2$ has exponents sequence 242 and degree $2 + 4 + 2 = 8$.

A formal power series $f(\mathbf{x})$ is *quasisymmetric* if any two monomials with the same exponent sequence have the same coefficient.

Ex.
$f(\mathbf{x}) = 7x_1^4 + 7x_2^4 + \cdots - x_1^2 x_2 - x_1^2 x_3 - \cdots = 7\sum_i x_i^4 - \sum_{i<j} x_i^2 x_j$.

The *algebra of quasisymmetric functions*, QSym = QSym($\mathbf{x}$), is the set of all $f(\mathbf{x})$ which are quasisymmetric of bounded degree. A basis for QSym is given by Gessel's *fundamental quasisymmetric functions* $F_{S,n}$ defined for each given n and $S \subseteq [n-1]$. If π is a permutation with $|\pi| = n$ and $\mathrm{Des}\,\pi = S$ we define $F_{\mathrm{Des}\,\pi} = F_{S,n}$.

Theorem (Gessel)

We have

$$F_{\mathrm{Des}\,\pi} F_{\mathrm{Des}\,\sigma} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} F_{\mathrm{Des}\,\tau}.$$

For the formula to be well defined, Des must be shuffle-compatible.

Given a linear permutation statistic st, we define an equivalence relation, $\sim$, on permutations by letting $\pi \sim \sigma$ if $|\pi| = |\sigma|$ and $\mathrm{st}\,\pi = \mathrm{st}\,\sigma$. Denote the equivalence class of π by $\mathsf{cl}\,\pi$.

Ex. If $\mathrm{st} = \mathrm{des}$ then $132 \sim 846$ since $|132| = 3 = |846|$ and $\mathrm{des}\,132 = 1 = \mathrm{des}\,846$. Also $\mathsf{cl}\,132 = \{132, 213, \ldots, 846, \ldots\}$.

Call st a *descent statistic* if for any permutations π, σ with $|\pi| = |\sigma|$ and $\mathrm{Des}\,\pi = \mathrm{Des}\,\sigma$ we have $\mathrm{st}\,\pi = \mathrm{st}\,\sigma$.

Ex. We have des is a descent statistic because if $\mathrm{Des}\,\pi = \mathrm{Des}\,\sigma$ then $\mathrm{des}\,\pi = |\,\mathrm{Des}\,\pi| = |\,\mathrm{Des}\,\sigma| = \mathrm{des}\,\sigma$.

Note that if st is a descent statistic then for any permutations π, σ we have $|\pi| = |\sigma|$ and $\mathrm{Des}\,\pi = \mathrm{Des}\,\sigma$ implies $\mathsf{cl}\,\pi = \mathsf{cl}\,\sigma$.

Theorem (Gessel-Zhuang)

Let st be a descent statistic on linear permutations. Then st is shuffle-compatible if and only if there is an algebra A with basis $\{b_{\mathsf{cl}\,\pi}\}$ such that the map $\mathrm{QSym} \to A$ obtained by linearly extending $F_{\mathrm{Des}\,\pi} \mapsto b_{\mathsf{cl}\,\pi}$ is an algebra homomorphism.

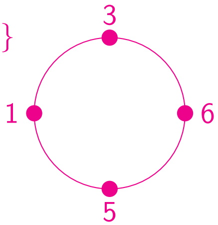
Gessel and Zhuang use this result to prove shuffle-compatibility of many permutation statistics by finding a corresponding algebra A.

A linear permutation $\pi = \pi_1\pi_2\ldots\pi_n$ of a set S has a corresponding *cyclic permutation*

$$[\pi] = \{\pi_1\pi_2\ldots\pi_n,\ \pi_2\ldots\pi_n\pi_1,\ \ldots,\ \pi_n\pi_1\ldots\pi_{n-1}\}.$$

Ex. $[3651] = \{3651, 6513, 5136, 1365\}$

$= [5136].$



A *cyclic statistic* is a function cst whose domain in all cyclic permutations. Cyclic statistics can be lifted from linear ones. The *cyclic descent set* of a linear permutation $\pi = \pi_1 \ldots \pi_n$ is

$$\operatorname{cDes}\pi = \{i \ : \ \pi_i > \pi_{i+1} \text{ where } i \text{ is taken modulo } n\},$$

and the *cyclic descent number* of π is

$$\operatorname{cdes}\pi = |\operatorname{cDes}\pi|.$$

Ex. If $\pi = 73698$ then $\operatorname{cDes}\pi = \{1, 4, 5\}$ and $\operatorname{cdes}\pi = 3$.

Note that $\emptyset \subset \operatorname{cDes}\pi \subset [n]$. Such a statistic is called *non-Escher*. Now define analogues for cyclic permutations by

$$\operatorname{cDes}[\pi] = \{\{\operatorname{cDes}\sigma \mid \sigma \in [\pi]\}\} \quad \text{and} \quad \operatorname{cdes}[\pi] = \operatorname{cdes}\pi.$$

Note $\operatorname{cdes}[\pi]$ is well defined since $\operatorname{cdes}\sigma$ is the same for all $\sigma \in [\pi]$.

Ex. $[1536] = \{1536, 5361, 3614, 6153\}$ so we have $\operatorname{cDes}[1536] = \{\{ \ \{2, 4\}, \ \{1, 3\}, \ \{2, 4\}, \ \{1, 3\} \ \}\}$ and $\operatorname{cdes}[1536] = 2$.

If $S \subseteq [n]$ and $i \in [n]$ then the *cyclic shift of S by i* is

$$S + i = \{s + i \pmod{n} \mid s \in S\}.$$

and let $[S] = \{S + i \mid i \in [n]\}$.
Ex. If $\pi = \pi_1 \dots \pi_n$ is a linear permutation of S then

$$\mathrm{cDes}[\pi] = \{\{\mathrm{cDes}\,\pi + i \mid i \in [n]\}\} = [\mathrm{cDes}\,\pi].$$

Cyclic permutations $[\pi], [\sigma]$ with $\pi \cap \sigma = \emptyset$ have *cyclic shuffle set*

$$[\pi] \sqcup [\sigma] = \{[\tau] \mid \tau = \pi' \sqcup \sigma' \text{ where } \pi' \in [\pi], \sigma' \in [\sigma]\}.$$

Statistic cst is *cyclic shuffle-compatible* if the multiset
$\mathrm{cst}([\pi] \sqcup [\sigma])$ depends only on $\mathrm{cst}[\pi], \mathrm{cst}[\sigma], |\pi|$, and $|\sigma|$.

Theorem (Adin, Gessel, Reiner, and Roichman)

Both cDes *and* cdes *are cyclic shuffle-compatible.* □

Adin, Gessel, Reiner and Roichman defined a cyclic analogue of the fundamental quasisymmetric functions, $F_{[S],n}$, for non-Escher subsets S of $[n]$. The algebra generated by the $F_{[S]}$ is denoted cQSym^-. They needed cyclic shuffle-compaitibility to prove that a certain formula for multiplying these functions was well defined. Domagalski, Liang, Minnich, S, Schmidt, and Sietema found a combinatorial way to lift linear shuffle-compatibility result to the cyclic realm. Liang, S, and Zhuang have shown how cyclic shuffle-compatibility could be proved algebraically.

Theorem (Liang-S-Zhuang)

Let cst *be a cyclic descent statistic. Then* cst *is cyclic shuffle-compatible if and only if there is an algebra* C *with basis* $\{b_{\mathrm{cl}[\pi]}\}$ *such that the map* $\mathrm{cQSym}^- \to C$ *obtained by linearly extending* $F_{\mathrm{cDes}[\pi]} \mapsto b_{\mathrm{cl}[\pi]}$ *is an algebra homomorphism.*

Our other results include the following.

1. Proofs of cyclic shuffle-compatibility of cDes, cdes, cPk, cpk, Des, des, Pk, cpk, epk, eval, (cpk, cdes), (val, des), (cval, cdes), and (epk, des).

2. Explicit descriptions of the cyclic shuffle algebras C for various statistics.

3. Formulation of a result for deriving cyclic shuffle-compatibility results from linear ones.

4. Description of various equivalences and symmetries between statistics.

The *inversion number* of a linear permutation $\pi = \pi_1 \dots \pi_n$ is

$$\operatorname{inv} \pi = \text{number of copies of the pattern 21 in } \pi.$$

The inversion number is not shuffle-compatible. Given a permutation π let $\overline{\pi}$ be the corresponding consecutive pattern. Let $\overline{\Pi}$ be a set of consecutive patterns. Define a statistic on permutations σ by

$$\operatorname{st}_{\overline{\Pi}}(\sigma) = \text{number of copies of a } \overline{\pi} \in \overline{\Pi} \text{ in } \sigma.$$

Note that

$$\operatorname{st}_{\overline{21}}(\sigma) = \operatorname{des} \sigma$$

and

$$\operatorname{st}_{\{\overline{132},\overline{231}\}}(\sigma) = \operatorname{pk} \sigma,$$

the number of peaks of σ. Both des and pk are shuffle compatible.

Question

For what $\overline{\Pi}$ is $\operatorname{st}_{\overline{\Pi}}$ shuffle compatible?

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THANKS FOR LISTENING!

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