Cyclic Shuffle Compatibility

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Linear shuffle compatibility

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Remarks and open problem

Let $\mathbb{P} = \{1, 2, 3, \dots\}$ and let $S \subset \mathbb{P}$ be finite. A *(linear) permutation* of S is a linear ordering $\pi = \pi_1 \pi_2 \dots \pi_n$ of the elements of S. Let

$$L(S) = \{\pi \ : \ \pi \text{ a permutation of } S\}.$$

Ex. $L(\{2, 4, 7\}) = \{247,\ 274,\ 427,\ 472,\ 724,\ 742\}$.

A *statistic* is a function st whose domain is $\uplus_S L(S)$. Examples include the *Descent set* of π which is

$$\mathrm{Des}\,\pi = \{i \ : \ \pi_i > \pi_{i+1}\}.$$

The *descent number* and *major index* of π are

$$\mathrm{des}\,\pi = |\,\mathrm{Des}\,\pi| \quad \text{and} \quad \mathrm{maj}\,\pi = \sum_{i \in \mathrm{Des}\,\pi} i$$

where $|\cdot|$ is cardinality. Say st is a *descent statistic* if $\mathrm{st}\,\pi$ only depends on $\mathrm{Des}\,\pi$ and $|\pi|$, for example, des and maj.

Ex. If $\pi = 53698$ then

$$\mathrm{Des}\,\pi = \{1, 4\}, \quad \mathrm{des}\,\pi = 2, \quad \mathrm{maj}\,\pi = 1 + 4 = 5.$$

If π, σ are permutations with $\pi \cap \sigma = \emptyset$ then their *shuffle set* is

$$\pi \sqcup\!\sqcup\, \sigma = \{\tau \ : \ |\tau| = |\pi| + |\sigma| \text{ and } \pi, \sigma \text{ are subwords of } \tau\}.$$

Ex. We have

$$25 \sqcup\!\sqcup\, \mathbf{74} = \{25\mathbf{74},\ 2\mathbf{7}5\mathbf{4},\ 2\mathbf{74}5,\ \mathbf{7}25\mathbf{4},\ \mathbf{7}2\mathbf{4}5,\ \mathbf{74}25\}.$$

Statistic st is *shuffle compatible* if the multiset $\mathrm{st}(\pi \sqcup\!\sqcup\, \sigma)$ depends only on $\mathrm{st}\,\pi, \mathrm{st}\,\sigma$ and $|\pi|, |\sigma|$.

Ex. One can show that des is shuffle compatible. For example

$$\mathrm{des}(25 \sqcup\!\sqcup\, 74) = \{\{1,1,1,1,2,2\}\} = \mathrm{des}(12 \sqcup\!\sqcup\, 43).$$

Shuffle compatibility is implicit in the work of Stanley on *P*-partitions. It is also needed to show that a formula for the multiplication of fundamental quasisymmetric functions is well defined. It was explicitly defined and studied by Gessel and Zhuang. Further work was done by Grinberg, by Oğuz, and by Baker-Jarvis and S.

Recently Adin, Gessel, Reiner, and Roichman defined cyclic quasisymmetric functions and a cyclic notion of shuffling. A linear permutation $\pi = \pi_1\pi_2\ldots\pi_n$ has corresponding *cyclic permutation*

$$[\pi] = \{\pi_1\pi_2\ldots\pi_n,\ \pi_2\ldots\pi_n\pi_1,\ \ldots, \pi_n\pi_1\ldots\pi_{n-1}\}.$$

Ex. $[2547] = \{2547, 5472, 4725, 7254\}$.

Certain linear statistics can be lifted to the cyclic realm. Define the *cyclic descent number* of a linear π with $|\pi| = n$ to be

$$\operatorname{cdes}\pi = \#\{i\ :\ \pi_i > \pi_{i+1} \text{ where } i \text{ is taken modulo } n\}.$$

Also define the *cyclic descent number* of $[\pi]$ to be

$$\operatorname{cdes}[\pi] = \operatorname{cdes}\pi.$$

This is well defined since $\sigma, \sigma' \in [\pi]$ implies $\operatorname{cdes}\sigma = \operatorname{cdes}\sigma'$.

Ex. $\operatorname{cdes}(2547) = 2$ since $5 > 4$ and $7 > 2$. Similarly $\operatorname{cdes}(5472) = \operatorname{cdes}(4725) = \operatorname{cdes}(7254) = 2$. So $\operatorname{cdes}[2547] = 2$.

Cyclic permutations $[\pi], [\sigma]$ with $\pi \cap \sigma = \emptyset$ have *cyclic shuffle set*

$$[\pi] \sqcup\!\sqcup [\sigma] = \{[\tau] \ : \ \tau = \pi' \sqcup\!\sqcup \sigma' \text{ for } \pi' \in [\pi] \text{ and } \sigma' \in [\sigma]\}.$$

Ex.

$$[143] \sqcup\!\sqcup [\mathbf{25}] = \{[143\mathbf{25}], [143\mathbf{52}], [14\mathbf{2}3\mathbf{5}], [14\mathbf{5}3\mathbf{2}], [14\mathbf{25}3], [14\mathbf{52}3],$$
$$[1\mathbf{2}43\mathbf{5}], [1\mathbf{5}43\mathbf{2}], [1\mathbf{2}4\mathbf{5}3], [1\mathbf{5}4\mathbf{2}3], [1\mathbf{25}43], [1\mathbf{52}43]\}\,.$$

A cyclic permutation statistic cst is *cyclic shuffle compatible* if the multiset $\text{cst}([\pi] \sqcup\!\sqcup [\sigma])$ depends only on $\text{cst}[\pi], \text{cst}[\sigma]$, and $|\pi|, |\sigma|$. We have a method for lifting linear shuffle compatibility results to cyclic ones. For a finite $S \subset \mathbb{P}$ let

$$C(S) = \{[\pi] \ : \ [\pi] \text{ is a cyclic permutation of } S\}.$$

Let $S \subset \mathbb{P}$ and $m = \max S$. Define the *maximum removal map*

$$M : C(S) \to L(S \setminus \{m\})$$

by $M[\pi] = \pi'$ where π' is obtained from π by rotating m to the right end of π and removing m.

Ex. If $[\pi] = [32856]$ then rotation gives 56328 so $M[\pi] = 5632$.

Note that M is a bijection.

Lemma (Lifting Lemma)

Let cst *be a cyclic descent statistic and* st *be a shuffle compatible linear descent statistic such that the following conditions hold.*

(a) *For any* S *and* $[\tau], [\tau'] \in C(S)$

$$\mathrm{st}(M[\tau]) = \mathrm{st}(M[\tau']) \quad \textit{implies} \quad \mathrm{cst}[\tau] = \mathrm{cst}[\tau'].$$

(b) *Given any* $[\pi], [\pi']$ *such that* $\mathrm{cst}[\pi] = \mathrm{cst}[\pi']$, *...*

Then cst *is cyclic shuffle compatible.*

Lemma (Lifting Lemma)

Let cst *be a cyclic descent statistic and* st *be a shuffle compatible linear descent statistic such that the following conditions hold.*

(a) *For any S and $[\tau], [\tau'] \in C(S)$*

$$\mathrm{st}(M[\tau]) = \mathrm{st}(M[\tau']) \quad \textit{implies} \quad \mathsf{cst}[\tau] = \mathsf{cst}[\tau'].$$

(b) *Given any $[\pi], [\pi']$ such that $\mathsf{cst}[\pi] = \mathsf{cst}[\pi']$, ...*

Then cst *is cyclic shuffle compatible.*

Corollary

The statistic cdes *is cyclic shuffle compatible.*

Proof.

We show that (a) is satisfied with $\mathrm{st} = \mathrm{des}$. If $M[\tau] = \sigma$ then $\mathsf{cdes}[\tau] = 1 + \mathrm{des}\,\sigma$ since $m = \max S$ causes a cyclic descent in $[\tau]$ which does not correspond to a descent of σ. It follows that

$$\mathsf{cdes}[\tau] = 1 + \mathrm{des}(M[\tau]) = 1 + \mathrm{des}(M[\tau']) = \mathsf{cdes}[\tau']. \qquad \square$$

(1) Other statistics. One can use the Lifting Lemma to show that the cyclic statistics corresponding to the descent set and also peak set and peak number are cyclic shuffle compatible.

(2) Cyclic patterns. There has been recent work on cyclic pattern containment and avoidance including, for example, a cyclic version of the Erdős-Szekeres Theorem. See Jinting Liang's talk for details.

(3) Cyclic maj? That maj is shuffle compatible is implied by the following result of Stanley:

$$\sum_{\tau \in \pi \sqcup\!\sqcup \sigma} q^{\mathrm{maj}\,\tau} = q^{\mathrm{maj}\,\pi + \mathrm{maj}\,\sigma} \left[\begin{array}{c} |\pi| + |\sigma| \\ |\pi| \end{array} \right]_q$$

where $\left[\begin{array}{c} \cdot \\ \cdot \end{array} \right]$ is a q-binomial coefficient.

Is there a cyclic version of maj satisfying a similar equation?

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THANKS FOR
LISTENING!