Counting permutations by congruence class of major index

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The major index

The inversion number

Shuffles

The case $k = \ell$

Outline

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Theorem

If q is an indeterminate then

$$\sum_{\pi \in S_n} q^{\operatorname{maj} \pi} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}). \qquad \blacksquare$$

Given k, ℓ we let $m_n^{k,\ell}$ be the $k \times \ell$ matrix with (i,j) entry

$$m_n^{k,\ell}(i,j) = \#\{\pi \in S_n \;:\; \operatorname{maj} \pi \equiv i \pmod{k},\ \operatorname{maj} \pi^{-1} \equiv j \pmod{\ell}\}.$$

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(1) Prove the special case $k = 1$: $\qquad m_n^{1,\ell}(i,j) = n!/\ell \quad \forall i,j.$

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(3) Use (2) and induction on n to prove the final case $n > k$.

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$$\operatorname{inv}\pi = \#\{(a_i, a_j) \;:\; i < j \quad \text{and} \quad a_i > a_j\}.$$

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We say maj and inv are equidistributed, i.e., have the same generating function. So are (maj, imaj) and (inv, imaj).

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So

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Outline

If $\tau = a_1 a_2 \ldots a_n$ is a sequence and $I = \{i_1, i_2, \ldots, i_\ell\}$ then

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If $\#\pi = \#I = \ell$ and $\#\sigma = m$ then the *I-shuffle* of π and σ is

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Ex. If $\ell = 4$ and $\tau = 3\,6\,1\,4\,7\,2\,5$ then $\tau = 3\,1\,4\,2 \sqcup\!\sqcup_I 6\,7\,5$. So

$$\tau' = 3\,1\,4\,2 \sqcup\!\sqcup_{I+1} 6\,7\,5 = 6\,3\,7\,1\,4\,5\,2.$$

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Note. (a) f is bijective: for f^{-1} use $l - 1$.

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$$\tau = \pi \bigsqcup_{l} \sigma \quad \text{imples} \quad \tau' = \pi \bigsqcup_{l+1} \sigma \quad \text{for} \quad \pi \in S_\ell.$$

Ex. If $\ell = 4$ and $\tau = 3\ 6\ 1\ 4\ 7\ 2\ 5$ implies $\tau' = 6\ 3\ 7\ 1\ 4\ 5\ 2$.

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But $\gcd(n, \ell) = 1$, so iterating f gives a bijection

$$M(i,j) \longleftrightarrow M(i+1, j). \qquad \blacksquare$$

Outline

The major index

The inversion number

Shuffles

The case $k = \ell$

Theorem

If $\gcd(k,\ell) = 1$ *and* $d \geq 1$ *with* $kd, \ell d \leq n$ *then* $m_n^{kd,\ell d}$ *is composed of* $d \times d$ *blocks all equal to*

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Let μ and ϕ be the number-theoretic Möbius and Euler functions, respectively.

Theorem

If $1 \leq i,j \leq n$ *then*

$$m_n^{n,n}(i,j) = \frac{1}{n^2} \sum_{d|n} d^{n/d} \left(\frac{n}{d}\right)! \; \phi(d)^2 \; \frac{\mu\left(\frac{d}{\gcd(i,d)}\right) \mu\left(\frac{d}{\gcd(j,d)}\right)}{\phi\left(\frac{d}{\gcd(i,d)}\right) \phi\left(\frac{d}{\gcd(j,d)}\right)}.$$

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By induction on n we can prove the following result.

Theorem

For each prime p, there are sequences $(q_n)_{n\geq 1}$, $(r_n)_{n\geq 1}$, *and* $(s_n)_{n\geq 1}$ *such that*

$$m_{np}^{p,p} = \begin{bmatrix} q_n \, J_{1,1} & r_n \, J_{1,p-1} \\ r_n \, J_{p-1,1} & s_n \, J_{p-1,p-1} \end{bmatrix}. \qquad \blacksquare$$