Combinatorial and Algebraic Approaches to Lucas Analogues

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Motivation and the Lucas sequence

Lucasnomials combinatorially (with Bennet, Carrillo, Machacek)

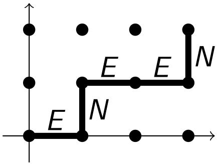
Lucasnomials algebraically (with Tirrell à la Stanley)

Comments and open problems

For integers $0 \leq k \leq n$ the corresponding *binomial coefficient* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \overset{?}{\in} \mathbb{Z}. \tag{1}$$

A. Give a combinatorial interpretation to $\binom{n}{k}$.
Interpretation 1. $\binom{n}{k} = \#$ of k-element subsets of $\{1, \ldots, n\}$.
Interpretation 2. Consider paths p in the integer lattice $\mathbb{Z}^2$ using unit steps E (add the vector $(1,0)$) and N (add the vector $(0,1)$).

$p =$ 

The number of paths p from $(0,0)$ to (m,n) is $\binom{m+n}{m}$ because p has $m+n$ total steps of which m must be E (and then the rest N).

B. Factor the top and bottom of (1) into primes and show that all primes in the denominator cancel into the numerator.

Let s and t be variables. The corresponding *Lucas sequence* is defined inductively by $\{0\} = 0$, $\{1\} = 1$, and

$$\{n\} = s\{n-1\} + t\{n-2\}$$

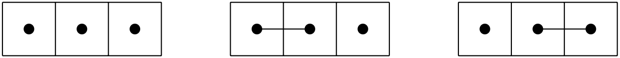
for $n \geq 2$. For example,

$$\{2\} = s, \ \{3\} = s^2 + t, \ \{4\} = s^3 + 2st.$$

We have the following specializations.
(1) $s = t = 1$ implies $\{n\} = F_n$, the Fibonacci numbers.
(2) $s = 2$, $t = -1$ implies $\{n\} = n$.
(3) $s = 1 + q$, $t = -q$ implies $\{n\} = 1 + q + \cdots + q^{n-1} = [n]_q$.
So when proving theorems about the Lucas sequence, one gets results about the Fibonacci numbers, the nonnegative integers, and q-analogues for free.

The *Lucas analogue* of $\prod_i n_i / \prod_j k_j$ is $\prod_i \{n_i\} / \prod_j \{k_j\}$. When is the Lucas analogue a polynomial in s, t? If so, is there a combinatorial interpretation? Given a row of n squares, let $\mathcal{T}(n)$ be the set of all tilings of the row with dominoes and monominoes.

$\mathcal{T}(3)$:



The *weight* of a tiling T is

$$\mathrm{wt}\ T = s^{\text{number of monominoes in } T}\ t^{\text{number of dominoes in } T}.$$

Similarly, given any set of tilings $\mathcal{T}$ we define its *weight* to be

$$\mathrm{wt}\,\mathcal{T} = \sum_{T \in \mathcal{T}} \mathrm{wt}\ T.$$

To illustrate $\mathrm{wt}(\mathcal{T}(3)) = s^3 + 2st = \{4\}$.

Theorem

For all $n \geq 1$ we have $\{n\} = \mathrm{wt}(\mathcal{T}(n-1))$.

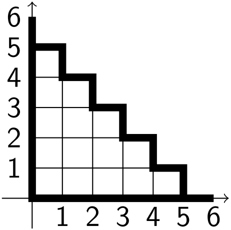
Previous work on the Lucas analogue of the binomial coefficients was done by Benjamin-Plott and Savage-Sagan.

Given $0 \le k \le n$ the corresponding *Lucasnomial* is

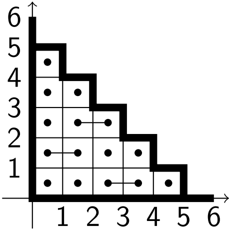
$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

where $\{n\}! = \{1\}\{2\}\ldots\{n\}$. This is a polynomial in s, t. Consider the *staircase* δ_n in the first quadrant of $\mathbb{R}^2$ consisting of a row of $n-1$ unit squares on the bottom, then $n-2$ one row above, etc.

δ_6:



a tiling:

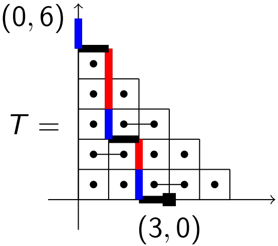


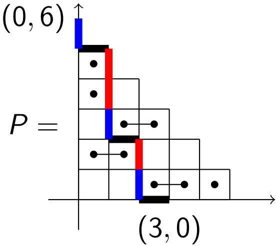
The set of *tilings of δ_n* is $\mathcal{T}(\delta_n)$ consisting of all tilings of the rows of δ_n. Using the combinatorial interpretation of $\{n\}$ we see

$$\mathrm{wt}\, \mathcal{T}(\delta_n) = \{n\}!$$

Theorem For $0 \le k \le n$ we have $\left\{ {n \atop k} \right\}$ is a polynomial in s, t.

Proof sketch. It suffices to construct a partition of $\mathcal{T}(\delta_n)$ such that $\{k\}!\{n-k\}!$ divides $\mathrm{wt}\, B$ for all blocks B of the partition. Given $T \in \mathcal{T}(\delta_n)$ we will find the B containing T as follows. Construct a lattice path p in T going from $(k, 0)$ to $(0, n)$ and using unit steps N (north) and W (west) by: move N if possible without crossing a domino or leaving δ_n; otherwise move W. If $n = 6$ and $k = 3$, and





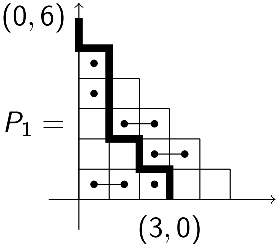
An N step just after a W is an *NL* step; otherwise it is an *NI* step. B is all tilings with path p that have the same tiles as T in all squares to the right of each *NL* step and in all squares to the left of each *NI* step. This gives a *partial tiling*, P. The variable parts of P contribute $\{k\}!\{n-k\}!$. □

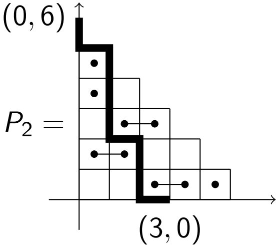
Proposition $\left\{ {n \atop k} \right\} = \{k+1\} \left\{ {n-1 \atop k} \right\} + t\{n-k-1\} \left\{ {n-1 \atop k-1} \right\}.$

Proof. From the previous proof we have

$$\left\{ {n \atop k} \right\} = \sum_P \mathrm{wt}\, P$$

where the sum is over the fixed tiles in all partial tilings P of δ_n whose path begins at $(k, 0)$. If the path p of P begins with an N step then the tiling to its left contributes $\{k+1\}$ and the rest of p contributes $\left\{ {n-1 \atop k} \right\}$. If p begins with WN then the tiling to its right contributes $t\{n-k-1\}$ and the rest of p contributes $\left\{ {n-1 \atop k-1} \right\}$. □





Define the sequence of *Lucas atoms*, $P_n = P_n(s,t)$, inductively by

$$\prod_{d|n} P_d = \{n\}.$$

As examples $\{1\} = P_1$ so $P_1 = 1$. Also, $\{2\} = P_1 P_2 = P_2$. In general, if p is prime then $P_p = \{p\}$. When $n = 6$

$$P_6 = \frac{\{6\}}{P_1 P_2 P_3} = \frac{s^5 + 4s^3 t + 3st^2}{s(s^2+t)} = s^2 + 3t.$$

Theorem

(i) *For all n we have $P_n(s,t) \in \mathbb{N}[s,t]$ where $\mathbb{N} = \{0,1,2,\ldots\}$.*

(ii) *$\prod_n \{n\} / \prod_k \{k\}$ is a polynomial if and only if, after expressing each factor as a product of atoms, all atoms in the denominator cancel. In this case, the quotient is in $\mathbb{N}[s,t]$.* □

Theorem

For all $0 \le k \le n$ we have $\left\{ n \atop k \right\} \in \mathbb{N}[s,t]$.

Proof.

By the previous theorem it suffices to show, using $\{n\} = \prod_{d|n} P_d$, that the number of factors of P_d in the numerator is at least as great as the number in the denominator for all d. Now P_d is a factor of $\{n\}$ if and only if $d|n$. So the number of P_d's dividing $\{n\}!$ is the floor function $\lfloor n/d \rfloor$. Similarly, the number of P_d's dividing $\{k\}!\{n-k\}!$ is $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor$. We are done since $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor \le \lfloor n/d \rfloor$. □

The *cyclotomic polynomials* $\Phi_n = \Phi_n(q)$ are defined inductively by

$$\prod_{d|n} \Phi_d(q) = q^n - 1.$$

Recall that $\{n\}_{q+1,-q} = 1 + q + \cdots + q^{n-1} = (q^n - 1)/(q-1)$.

Proposition

For all $n \ge 2$ we have $P_n(q+1, -q) = \Phi_n(q)$. □

There are Lucas analogues of many results about cyclotomic polynomials.

Theorem (Gauss)

If $n \geq 5$ is square-free and satisfies $n \equiv 1 \pmod 4$, then there are polynomials $A_n(q)$ and $B_n(q)$. such that

$$4\Phi_n(q) = A_n^2(q) - (-1)^{(n-1)/2} nq^2 B_n^2(q)$$

where $A_n(q), B_n(q) \in \mathbb{Z}[q]$ are palindromic. □

Theorem (S and Tirrell)

If $n \geq 5$ is square-free and satisfies $n \equiv 1 \pmod 4$, then there are polynomials $E_n(s,t)$ and $F_n(s,t)$. such that

$$4P_n(s,t) = E_n^2(s,t) - nt^2 F_n^2(s,t)$$

where $E_n(s,t), F_n(s,t) \in \mathbb{Z}[s,t]$. □

The proof of thie Lucas analogue of Gauss' Theorem uses gamma expansions. A polynomial $p(q) = \sum_{i \geq 0} c_i q^i$ has *total degree*

$$\operatorname{tdeg} p(q) = k + l$$

where k, l are the smallest and largest indices with $c_k \neq 0$ and $c_l \neq 0$, respectively. Call $p(q)$ with $\operatorname{tdeg} p(q) = d$ *palindromic* if

$$c_i = c_{d-i}$$

for $0 \leq i \leq d$. If $p(q)$ is palindromic then its *gamma expansion* is

$$p(q) = \gamma_0 (1+q)^d + \gamma_1 (1+q)^{d-2} q + \cdots = \sum_{i \geq 0} \gamma_i (1+q)^{d-2i} q^i$$

Example. $p(q) = q + 7q^2 + 7q^3 + q^4$ has $\operatorname{tdeg} p(q) = 1 + 4 = 5$.
$p(q)$ is palindromic: $c_0 = c_5 = 0$, $c_1 = c_4 = 1$, $c_2 = c_3 = 7$.
$p(q) = 0 \cdot (1+q)^5 + 1 \cdot (1+q)^3 q + 4 \cdot (1+q) q^2$.

It is easy to see either inductively or combinatorially that

$$\{n\} = \gamma_0 s^{n-1} + \gamma_1 s^{n-3} t + \gamma_2 s^{n-5} t^2 + \cdots$$

for coefficients $\gamma_i \geq 0$. So

$$[n]_q = \{n\}_{1+q,-q} = \gamma_0 (1+q)^{n-1} - \gamma_1 (1+q)^{n-3} q + \gamma_2 (1+q)^{n-5} q^2 - \cdots$$

which is the gamma expansion of $[n]_q$. From $\{n\} = \prod_d P_d$ it follows that P_d can be written in the same form as $\{n\}$. So any Lucas analogue of a quotient of products can be written in this form as well. And substiting $s = 1 + q$, $t = -q$ gives the gamma expansion of the corresponding q-analogue which must be a palindrome. This makes it possible to lift the palindromes in Gauss' Theorem to the polynomials in s, t in our result.

1. Other combinatorial constants. Given any finite irreducible Coxeter group, the Lucas analogous of the Fuss-Catalan number $\mathrm{Cat}^k(W)$ and the Fuss-Narayana numbers $\mathrm{Nar}^k(W,i)$ are in $\mathbb{N}[s,t]$. For example, in type A the analogue is

$$\mathrm{Cat}^1\{A_{n-1}\} = \frac{1}{\{n+1\}}\left\{\begin{matrix}2n\\ n\end{matrix}\right\}.$$

Given a, b relatively prime positive integers we have the Lucas analogue of the corresponding rational Catalan number

$$\mathrm{Cat}\{a,b\} = \frac{1}{\{a+b\}}\left\{\begin{matrix}a+b\\ a\end{matrix}\right\}.$$

analogue	combinatorial proof	algebraic proof
$\mathrm{Cat}^k\{W\}$, $W = A$ – D	Y	Y
$\mathrm{Cat}^k\{W\}$, $W = E$ – I	?	Y
$\mathrm{Nar}^k\{W,i\}$, all W	Y*: $W = A$,$k = 1/?$	Y
$\mathrm{Cat}\{a,b\}$	?	Y

*Nenashev

2. Combinatorics of the P_n. Even though we know $P_n \in \mathbb{N}[s,t]$ for all n, we have no combinatorial interpretation for the its coefficients in general.

Proposition

If p is prime then

$$P_p = \sum_{k \geq 0} \binom{p-k-1}{k} s^{p-2k-1} t^k.$$

and

$$P_{2p} = \sum_{k \geq 0} \left[\binom{p-k}{k} + \binom{p-k-1}{k-1} \right] s^{p-2k-1} t^k. \quad \square$$

If a combinatorial interpretation can be found, it would be interesting to give a combinatorial proof of

$$\prod_{d \mid n} P_d = \{n\}.$$

3. Unimodality. A polynomial $p(q) = \sum_{i \geq 0} c_i q^i$ is *unimodal* if there is some index m such that

$$c_0 \leq c_1 \leq \cdots \leq c_m \geq c_{m+1} \geq \ldots$$

If $p(q)$ is palindromic and has nonegative coefficients in its gamma expansion, then $p(q)$ is unimodal. A Lucas analogue of a quotient of products has alternating gamma coefficients. Is it possible to use sign-reversing involutions to prove that some of these Lucas analogues are unimodal? This has been studied in a paper of Brittenham, Carroll, Petersen, and Thomas but only successfully on one example.

THANKS FOR

LISTENING!