Combinatorial Interpretations for Lucas Analogues

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

joint with Curtis Bennett, Juan Carrillo, and John Machacek

October 4, 2018

Motivation and the Lucas sequence

Binomial coefficient analogue

Catalan numbers and Coxeter groups

Comments and open problems

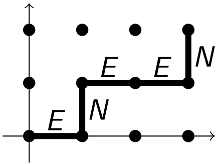
For integers $0 \leq k \leq n$ the corresponding *binomial coefficient* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

It is *not* obvious from this definition that this is an integer. It becomes obvious if we give a combinatorial interpretation to $\binom{n}{k}$.

Interpretation 1. $\binom{n}{k} = \#$ of k-element subsets of $\{1, \ldots, n\}$.

Interpretation 2. Consider paths p in the integer lattice $\mathbb{Z}^2$ using unit steps E (add the vector $(1,0)$) and N (add the vector $(0,1)$).

$p =$ 

The number of paths p from $(0,0)$ to (m,n) is $\binom{m+n}{m}$ because p has $m+n$ total steps of which m must be E (and then the rest N).

Let s and t be variables. The corresponding *Lucas sequence* is defined inductively by $\{0\} = 0$, $\{1\} = 1$, and

$$\{n\} = s\{n-1\} + t\{n-2\}$$

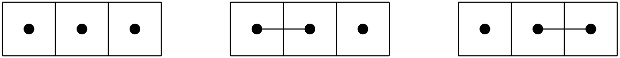
for $n \geq 2$. For example,

$$\{2\} = s, \ \{3\} = s^2 + t, \ \{4\} = s^3 + 2st.$$

We have the following specializations.
(1) $s = t = 1$ implies $\{n\} = F_n$, the Fibonacci numbers.
(2) $s = 2$, $t = -1$ implies $\{n\} = n$.
(3) $s = 1 + q$, $t = -q$ implies $\{n\} = 1 + q + \cdots + q^{n-1} = [n]_q$.
So when proving theorems about the Lucas sequence, one gets results about the Fibonacci numbers, the nonnegative integers, and q-analogues for free.

The *Lucas analogue* of $\prod_i n_i / \prod_j k_j$ is $\prod_i \{n_i\} / \prod_j \{k_j\}$. When is the Lucas analogue a polynomial in s, t? If so, is there a combinatorial interpretation? Given a row of n squares, let $\mathcal{T}(n)$ be the set of all tilings of the row with dominoes and monominoes.

$\mathcal{T}(3)$:



The *weight* of a tiling T is

$$\mathrm{wt}\ T = s^{\text{number of monominoes in } T}\ t^{\text{number of dominoes in } T}.$$

Similarly, given any set of tilings $\mathcal{T}$ we define its *weight* to be

$$\mathrm{wt}\,\mathcal{T} = \sum_{T \in \mathcal{T}} \mathrm{wt}\ T.$$

To illustrate $\mathrm{wt}(\mathcal{T}(3)) = s^3 + 2st = \{4\}$.

Theorem

For all $n \geq 1$ we have $\{n\} = \mathrm{wt}(\mathcal{T}(n-1))$.

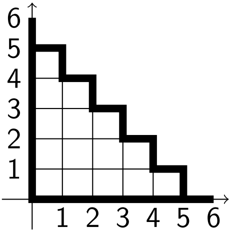
Previous work on the Lucas analogue of the binomial coefficients was done by Gessel-Viennot, Benjamin-Plott, Savage-Sagan.

Given $0 \le k \le n$ the corresponding *Lucasnomial* is

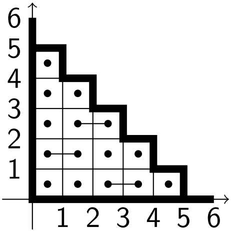
$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

where $\{n\}! = \{1\}\{2\}\ldots\{n\}$. This is a polynomial in s, t. Consider the *staircase* δ_n in the first quadrant of $\mathbb{R}^2$ consisting of a row of $n-1$ unit squares on the bottom, then $n-2$ one row above, etc.

δ_6:



a tiling:

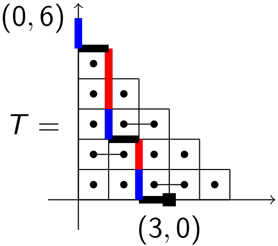


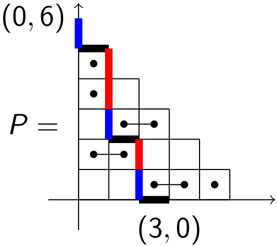
The set of *tilings of δ_n* is $\mathcal{T}(\delta_n)$ consisting of all tilings of the rows of δ_n. Using the combinatorial interpretation of $\{n\}$ we see

$$\mathrm{wt}\, \mathcal{T}(\delta_n) = \{n\}!$$

Theorem For $0 \le k \le n$ we have $\left\{ n \atop k \right\}$ is a polynomial in s, t.

Proof sketch. It suffices to construct a partition of $\mathcal{T}(\delta_n)$ such that $\{k\}!\{n-k\}!$ divides $\operatorname{wt} B$ for all blocks B of the partition. Given $T \in \mathcal{T}(\delta_n)$ we will find the B containing T as follows. Construct a lattice path p in T going from $(k, 0)$ to $(0, n)$ and using unit steps N (north) and W (west) by: move N if possible without crossing a domino or leaving δ_n; otherwise move W. If $n = 6$ and $k = 3$, and

$T =$ 

$P =$ 

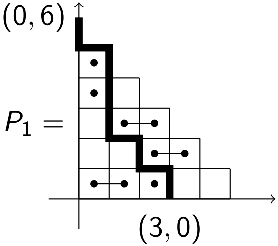
An N step just after a W is an *NL* step; otherwise it is an *NI* step. B is all tilings with path p and agreeing with T to the right of each *NL* step and to the left of each *NI* step. This gives a *partial tiling*, P. The variable parts of P contribute $\{k\}!\{n-k\}!$. □

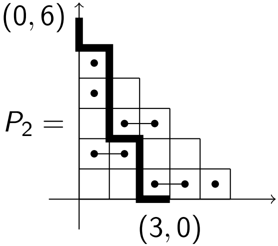
Proposition $\left\{ {n \atop k} \right\} = \{k+1\}\left\{ {n-1 \atop k} \right\} + t\{n-k-1\}\left\{ {n-1 \atop k-1} \right\}.$

Proof. From the previous proof we have

$$\left\{ {n \atop k} \right\} = \sum_P \mathrm{wt}\, P$$

where the sum is over the fixed tiles in all partial tilings P of δ_n whose path begins at $(k,0)$. If the path p of P begins with an N step then the tiling to its left contributes $\{k+1\}$ and the rest of p contributes $\left\{ {n-1 \atop k} \right\}$. If p begins with WN then the tiling to its right contributes $t\{n-k-1\}$ and the rest of p contributes $\left\{ {n-1 \atop k-1} \right\}$. □

$P_1 =$ 

$P_2 =$ 

For $n \geq 0$, the *Catalan numbers* are

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

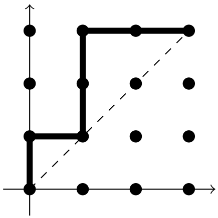
For example

n	0	1	2	3	4	5
C_n	1	1	2	5	14	42

Stanley has collected more than 200 combinatorial interpretations of C_n. One well-known interpretation is as follows.

Proposition

C_n is the nunber of lattice paths from $(0,0)$ to (n,n) using steps E and N and staying weakly above the line $y = x$.



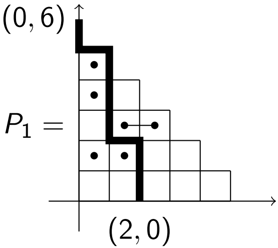
For $n \geq 0$ define the corresponding *Lucas-Catalan* to be

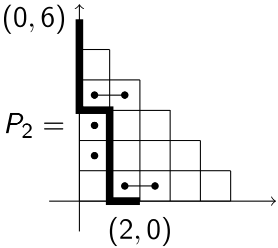
$$C_{\{n\}} = \frac{1}{\{n+1\}} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}.$$

Theorem For $n \geq 0$ we have $C_{\{n\}}$ is a polynomial in s, t.

Proof sketch. It suffices to construct a partition of $\mathcal{T}(\delta_{2n})$ such that $\{n\}!\{n+1\}!$ divides $\mathrm{wt}\, B$ for all blocks B. Given $T \in \mathcal{T}(\delta_{2n})$ we find the other tilings in B exactly as for $\left\{ \begin{smallmatrix} 2n \\ n-1 \end{smallmatrix} \right\}$ except that in the bottom row one lets both sides of the N step vary, always keeping the blocking domino if it is an NL step. □

Here are partial tilings corresponding to blocks for $C_{\{3\}}$, on the left for an NI step in the bottom row and on the right for an NL step.





The *finite Coxeter groups* W are those generated by reflections. Each irreducible W has *degree set* $D = \{d_1, \ldots, d_n\}$. The *Coxeter number* of W is $h = \max D$. The *Coxeter-Catalan number* of W is

$$\operatorname{Cat} W = \prod_{i=1}^{n} \frac{h + d_i}{d_i}.$$

$$\therefore \ \operatorname{Cat} A_n = \frac{(n+3)(n+4)\ldots(2n+2)}{(2)(3)\ldots(n+1)} = \frac{(2n+2)!}{(n+1)!(n+2)!} = C_{n+1}.$$

W	$d_1, \ldots, d_n$	h
A_n	$2, 3, 4, \ldots, n+1$	$n+1$
B_n	$2, 4, 6, \ldots, 2n$	$2n$
D_n	$2, 4, 6, \ldots, 2(n-1), n$	$2(n-1)$ (for $n \geq 3$)
E_6	$2, 5, 6, 8, 9, 12$	12
E_7	$2, 6, 8, 10, 12, 14, 18$	18
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	30
F_4	$2, 6, 8, 12$	12
H_3	$2, 6, 10$	10
H_4	$2, 12, 20, 30$	30
$I_2(m)$	$2, m$	m (for $m \geq 2$)

Define the *Lucas-Coxeter analogue*

$$\operatorname{Cat}\{W\} = \prod_{i=1}^{n} \frac{\{h+d_i\}}{\{d_i\}}.$$

Theorem

For all finite, irreducible W we have $\operatorname{Cat}\{W\}$ is a polynomial in s, t.

For $W = B_n$ we have

$$\operatorname{Cat}\{W\} = \frac{\{2n+2\}\{2n+4\}\ldots\{4n\}}{\{2\}\{4\}\ldots\{2n\}}.$$

For $0 \leq k \leq n$ and $d \geq 1$ define the *d-divisible Lucasnomial*

$$\left\{ \begin{matrix} n:d \\ k:d \end{matrix} \right\} = \frac{\{n:d\}!}{\{k:d\}!\{n-k:d\}!}$$

where $\{n:d\}! = \{d\}\{2d\}\ldots\{nd\}$.

Theorem

For all n, k, d we have $\left\{ \begin{matrix} n:d \\ k:d \end{matrix} \right\}$ is a polynomial in s, t.

1. Coefficients. Our proofs show our Lucas analogues are polynomials in s, t with coefficients in $\mathbb{N}$, the nonegative integers.

2. Fuss-Catalan numbers. The *Fuss-Catalan numbers* are, for $n \geq 0$ and $k \geq 1$,

$$C_{n,k} = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

Clearly $C_{n,1} = C_n$. Consider the Lucas analogue

$$C_{\{n,k\}} = \frac{1}{\{kn+1\}}\left\{ {(k+1)n \atop n} \right\}.$$

Theorem

For all n, k we have $C_{\{n,k\}}$ is a polynomial in s, t.

We can prove combinatorially that Fuss-Catalan Lucas analogues for the other infinite families of irreducible Coxeter goups are polynomials in $\mathbb{N}[s, t]$. Stanley-S have proved this algebraically for the exceptional Coxeter groups.

3. Rational Catalan numbers. Let $a, b \geq 1$ be relatively prime integers. The corresponding *rational Catalan number* is

$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a}$$

If $a = n$ and $b = n+1$ then

$$\mathrm{Cat}(a,b) = \frac{1}{2n+1}\binom{2n+1}{n} = C_n.$$

Theorem (Grossman (1950))

The number of lattice paths from $(0,0)$ to (a,b) using steps E and N and staying weakly above the line $y = (b/a)x$ is $\mathrm{Cat}(a,b)$.

Algebraically Bergeron et al. proved $\mathrm{Cat}\{a,b\}$ is a polynomial. Stanley-S. have shown algebraically it has coefficients in $\mathbb{N}$.

4. Narayana numbers. The *Narayana numbers* are

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Note $C_n = \sum_{k=1}^{n} N_{n,k}$. Stanley-S have shown algebraically that $N_{\{n,k\}}$ is a polynomial with coefficients in $\mathbb{N}$.

THANKS FOR LISTENING!