Stalking the Wild Fibonomial

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Binomial coefficients

Fibonomials

Comments

Let $\mathbb{Z}$ be the integers. Let $n, k \in \mathbb{Z}$ with $0 \leq k \leq n$. Define the *binomial coefficients* by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The $\binom{n}{k}$ can be displayed in *Pascal's triangle*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

It appears as if $\binom{n}{k}$ is always an integer. How can one prove this?

Induction. Extend $\binom{n}{k}$ to $k \in \mathbb{Z}$ by letting

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n.$$

Lemma

For $n \geq 1$ and $k \in \mathbb{Z}$ we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof.

Express the right side in terms of factorials and add. □

Theorem

We have $\binom{n}{k} \in \mathbb{Z}$ for all n, k.

Proof.

Induct on n. This is clear when $n = 0$. Assuming the result for $n - 1$ and using the lemma gives

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \text{ integer } + \text{ integer } = \text{ integer }.$$ □

Combinatorics. If $\binom{n}{k}$ is the cardinality of some set, then $\binom{n}{k} \in \mathbb{Z}$.
If S is a set then we let $\#S$ be the cardinality of S.

Theorem
If $\#S = n$ then

$$\binom{n}{k} = \#\{T \mid T \subseteq S \text{ and } \#T = k\}.$$

Factorization. If p is a prime and $n \in \mathbb{Z}$ then let

$$\nu_p(n) = \text{ the largest } k \text{ such that } p^k \text{ is a factor of } n.$$

Ex. Since $n = 50 = 2 \cdot 5^2$ we have $\nu_2(50) = 1$ and $\nu_5(50) = 2$.
If $c, d \in \mathbb{Z}$ then $c/d \in \mathbb{Z}$ iff $\nu_p(c) \geq \nu_p(d)$ for all primes p.

Theorem
For $n \geq 1$ and p prime we have

$$\nu_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

where $\lfloor \cdot \rfloor$ is the round-down function.

The *Fibonacci numbers* are defined by

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

Ex. The first few Fibonacci numbers are

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5.$$

A *Fibotorial* is

$$F_n^! = F_n F_{n-1} \cdots F_1.$$

A *Fibonomial* is

$$\binom{n}{k}_F = \frac{F_n^!}{F_k^! F_{n-k}^!}.$$

Ex.

$$\binom{5}{2}_F = \frac{F_5^!}{F_2^! F_3^!} = \frac{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1}{(1 \cdot 1)(2 \cdot 1 \cdot 1)} = 15.$$

We wish to show that $\binom{n}{k}_F$ is always an integer.

Induction. By induction on n it is easy to prove the following.

Lemma

For $n \geq k \geq 1$ we have

$$F_n = F_k F_{n-k+1} + F_{k-1} F_{n-k}.$$

Theorem

For $n \geq k \geq 1$ we have

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F.$$

Proof.

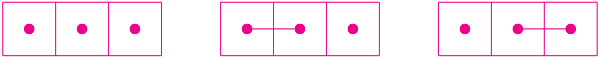
$$\begin{aligned}\binom{n}{k}_F &= \frac{F_n \cdot F^!_{n-1}}{F^!_k F^!_{n-k}} = \frac{(F_k F_{n-k+1} + F_{k-1} F_{n-k}) \cdot F^!_{n-1}}{F^!_k F^!_{n-k}} \\ &= F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F. \qquad \square\end{aligned}$$

Corollary

For $n \geq k \geq 0$ we have $\binom{n}{k}_F \in \mathbb{Z}$.

Combinatorics. Consider a row of n squares. A *tiling*, T, is a covering of the row with disjoint dominos (covering 2 squares) and monominos (covering 1 square). Let $\mathcal{T}_n$ be the set of such tilings.

Ex.

$\mathcal{T}_3$: 

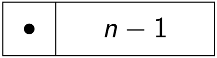
Note $\#\mathcal{T}_3 = 3 = F_4$.

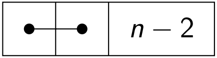
Theorem

For $n \geq 0$ we have:

$$\#\mathcal{T}_n = F_{n+1}.$$

Proof Induct on n. It's easy for $n = 0, 1$. For $n \geq 2$,

$\#\mathcal{T}_n$ = # 

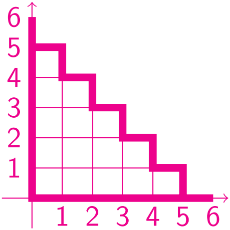
+ # 

$$
\begin{aligned}
&= \#\mathcal{T}_{n-1} + \#\mathcal{T}_{n-2} \\
&= F_n + F_{n-1} \quad \text{(by induction)} \\
&= F_{n+1}. \qquad \square
\end{aligned}
$$

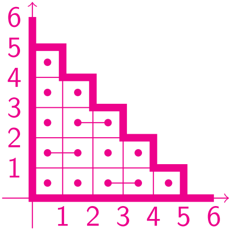
Consider the *staircase* δ_n in the first quadrant of $\mathbb{R}^2$ consisting of a row of $n-1$ unit squares on the bottom, then $n-2$ one row above, etc.

Ex.

δ_6:



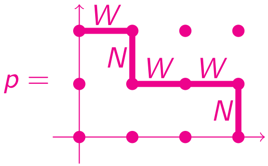
a tiling:



The set of *tilings of* δ_n is $\mathcal{T}(\delta_n)$ consisting of all tilings of the rows of δ_n. Using the combinatorial interpretation of F_n we see

$$\#\mathcal{T}(\delta_n) = \#\mathcal{T}_{n-1} \cdot \#\mathcal{T}_{n-2} \cdots = F_n \cdot F_{n-1} \cdots = F_n^!.$$

A *lattice path*, p, is a sequence of points in the integer lattice $\mathbb{Z}^2$.
Ex. The lattice path $p : (3,0),(3,1),(2,1),(1,1),(1,2),(0,2)$ is



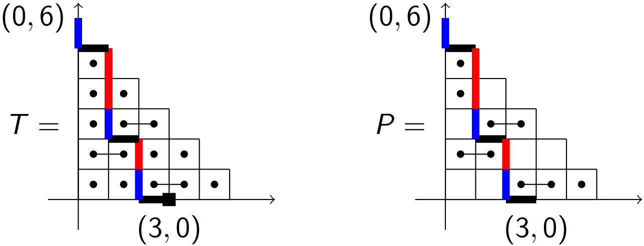
A *NW-lattice path* takes steps which are one unit north (add the vector $N = [0,1]$) or one unit west (add the vector $W = [-1,0]$). A NW-lattice path p from $(0,0)$ to $(-x,y)$ has $x+y$ steps. Choosing x of them to be W determines p since the other steps must by N by default. So the number of such paths is $\binom{x+y}{x}$.

A *partition* of a set S is a collection of disjoint nonempty subsets $\{B_1, B_2, \ldots, B_l\}$ called *blocks* whose union is S.
Ex. One parttion of $S = \{a,b,c,d,e,f\}$ is $\{\{a,c\},\{b,d,f\},\{e\}\}$.

The following combinatorial proof that $\binom{n}{k}_F$ is an integer was given by Bennett-Carrillo-Machacek-S. Earlier but less natural proofs were given by Benjamin-Plott, and S-Savage.

Theorem For $0 \le k \le n$ we have $\binom{n}{k}_F \in \mathbb{Z}$.

Proof. It suffices to construct a partition of $\mathcal{T}(\delta_n)$ such that $\#B = F_k^! F_{n-k}^!$ for all blocks B of the partition. Given $T \in \mathcal{T}(\delta_n)$ we will find the B containing T as follows. Construct a NW-lattice path p going from $(k, 0)$ to $(0, n)$: move N if possible without crossing a domino or leaving δ_n; otherwise move W. If $n = 6$ and $k = 3$, and



An N step just after a W is an *NL* step; otherwise it is an *NI* step. Let B be all tilings with path p and agreeing with T to the right of each NL step and to the left of each NI step, with associated *partial tiling*, P. The variable parts of P show $\#B = F_k^! F_{n-k}^!$. □

Proving $F_n = F_k F_{n-k+1} + F_{k-1} F_{n-k}$. This identity can be proved combinatorially by tiling.

Lucas polynomials. Let s, t be variables. The *Lucas polynomials*, $L_n = L_n(s, t)$ are defined by

$$L_0 = 0, \quad L_1 = 1, \quad L_n = sL_{n-1} + tL_{n-2} \text{ for } n \geq 2.$$

Ex. The first few Lucas polynomials are

$$L_0 = 0, \quad L_1 = 1, \quad L_2 = s, \quad L_3 = s^2 + t, \quad L_4 = s^3 + 2st.$$

Note that

$$L_n(1, 1) = F_n \quad \text{and} \quad L_n(2, -1) = n.$$

One can define Lucanomials in the obvious way and generalize all the results in this lecture.

Divisibility. The divisibility proof will not work directly for $\binom{n}{k}_F$. The *period modulo m* of the F_n is the smallest d such that $F_{n+d} \equiv F_n \pmod{m}$ for all sufficiently large n. The period always exists. It is a famous open problem to determine the period of F_n modulo p for all primes p. Using the $L_n(s, t)$ and cyclotomic polynomials, one can give a divisibility proof for the Lucanomials.

References.

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THANKS FOR LISTENING!