Stalking the Wild Fibonomial

Bruce Sagan
Department of Mathematics, Michigan State U.
East Lansing, MI 48824-1027, sagan@math.msu.edu
www.math.msu.edu/~sagan

and

Carla Savage
Department of Computer Science, North Carolina State U.
Raleigh, NC 27695-8206, savage@cayley.csc.ncsu.edu

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Fibonomials

A recursion

The combinatorial interpretation

Outline

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then integrality is obvious. (Here, "$\#$" denotes cardinality.)

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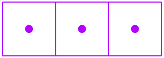

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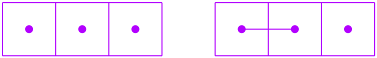

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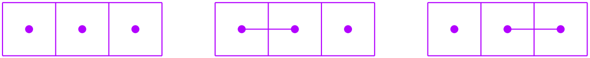

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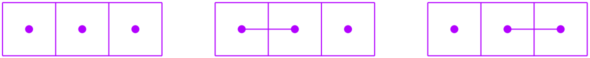

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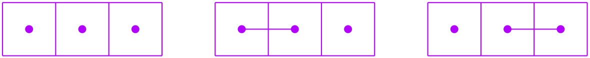

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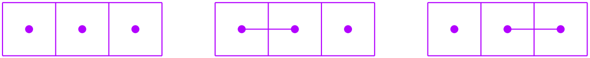

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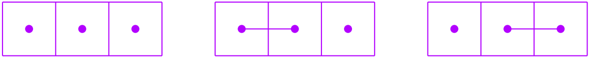

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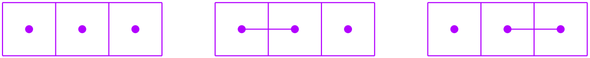

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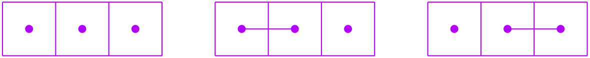

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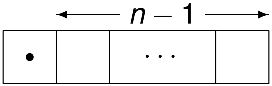
$\#\mathcal{T}_n \quad = \quad \#$ 

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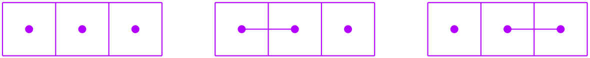

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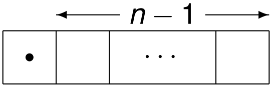

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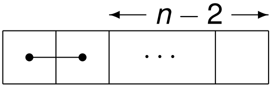

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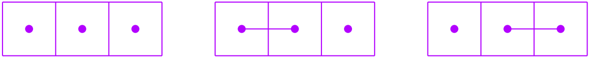

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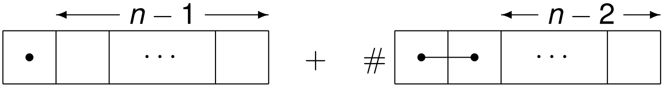

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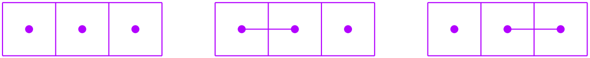

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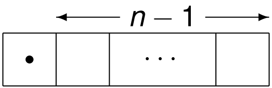
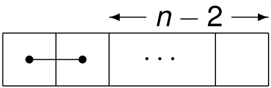
$\#\mathcal{T}_n = \#$  $+ \#$ 

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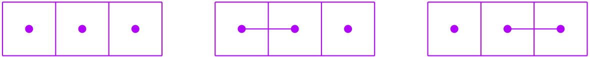

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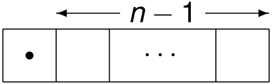
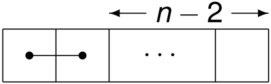
$\#\mathcal{T}_n = \#$ 

$+ \#$ 

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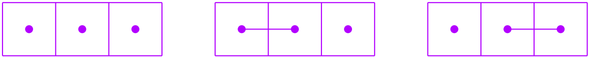
$\mathcal{T}_3$: 

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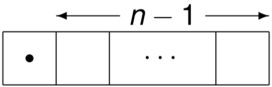
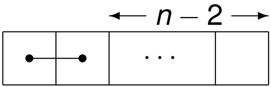
$\#\mathcal{T}_n = \#$ 

$+ \#$ 

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For $m, n \geq 1$ we have: $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$

Proof idea Use $F_{m+n} = \#\mathcal{T}_{m+n-1}$. $\square$

Theorem

For $m, n \geq 1$ we have:

$$\binom{m+n}{m}_F = F_{n+1}\binom{m+n-1}{m-1}_F + F_{m-1}\binom{m+n-1}{n-1}_F$$

Proof Using the definition of the fibonomials

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Outline

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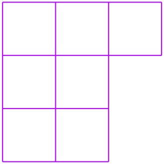
$\lambda = (3,2,2) =$ 

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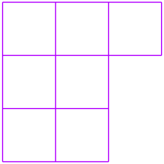
$\lambda = (3,2,2) =$ 

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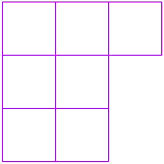
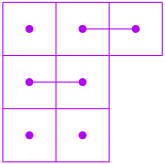
$\lambda = (3, 2, 2) =$ 

In $\mathcal{T}_\lambda$: 

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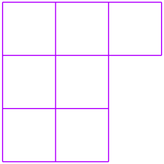
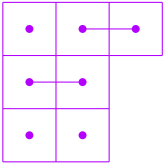
$\lambda = (3, 2, 2) =$ 

In $\mathcal{T}_\lambda$: 

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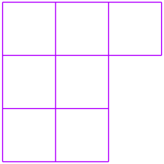
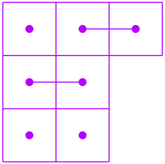
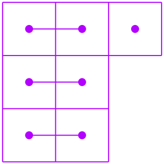
$\lambda = (3, 2, 2) =$ 

In $\mathcal{T}_\lambda$: 

In $\mathcal{D}_\lambda$: 

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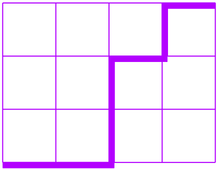

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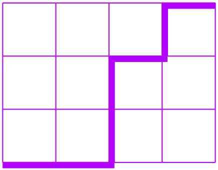

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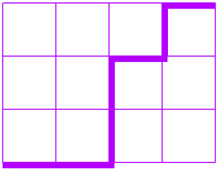

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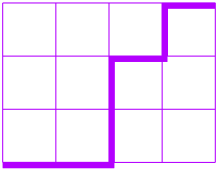

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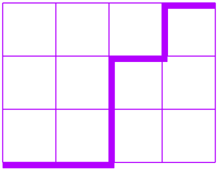

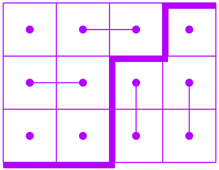

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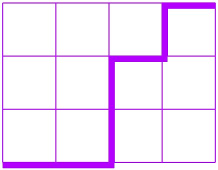

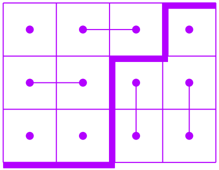

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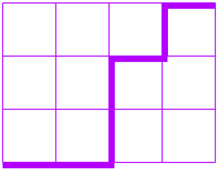

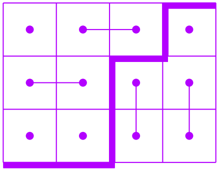

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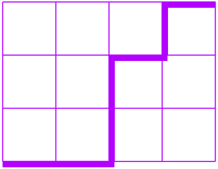

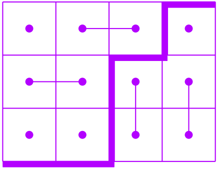

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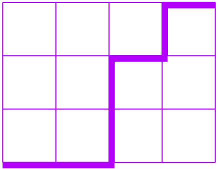

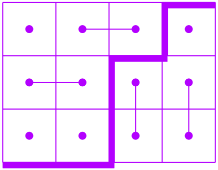

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THANKS FOR LISTENING!