Congruences and the Thue-Morse sequence

Emeric Deutsch
Department of Mathematics
Polytechnic University
Brooklyn, NY 11201, USA
deutsch@duke.poly.edu

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

1. Central binomial coefficients

2. Catalan numbers

3. Motzkin numbers

4. Open problems

1. Central binomial coefficients

Let

$$\rho_3(n) = \text{ remainder of } n \text{ on division by 3.}$$

Consider

n	0	1	2	3	4	5	6	7	8	9
$\rho_3\binom{2n}{n}$	1	2	0	2	1	0	0	0	0	2

Let

$$(n)_3 = n_l \ldots n_0, \text{ sequence of digits of } n \text{ base 3.}$$

Let

$$T(01) = \{n \;:\; (n)_3 \text{ has only zeros and ones}\}.$$

and

$$\omega_3(n) = \text{ number of ones in } (n)_3.$$

Benoit Cloitre and Reinhard Zumkeller conjectured the following.

Theorem 1 (D & S) *We have*

$$\binom{2n}{n} \equiv \left\{ \begin{array}{ll} (-1)^{\omega_3(n)} & \text{if } n \in T(01), \\ 0 & \text{else.} \end{array} \right\} \pmod 3$$

Theorem 2 (Lucas, 1877–8) *Let p be prime and let $(n)_p = n_l \ldots n_1 n_0$ and $(k)_p = k_l \ldots k_1 k_0$. Then*

$$\binom{n}{k} \equiv \binom{n_l}{k_l} \cdots \binom{n_1}{k_1}\binom{n_0}{k_0} \pmod{p}. \qquad \blacksquare$$

Corollary 3 *If there is a carry in computing the sum $(k)_p + (n-k)_p$ then*

$$\binom{n}{k} \equiv 0 \pmod{p}.$$

Proof Let i be the right-most place where there is a carry. So $k_i > n_i$ and $\binom{n_i}{k_i} = 0$. Now use Lucas. $\blacksquare$

Theorem 1 (D & S) *We have*

$$\binom{2n}{n} \equiv \left\{ \begin{array}{ll} (-1)^{\omega_3(n)} & \text{if } n \in T(01), \\ 0 & \text{else.} \end{array} \right\} \pmod{3}$$

Proof If there is a 2 in $(n)_3$ then there is a carry in $(n)_3 + (n)_3$ so $\binom{2n}{n} \equiv 0 \pmod{3}$ by the Corollary.

Otherwise $n \in T(01)$. So $n_i = 1$ iff the ith digit of $2n$ is 2. So by Lucas

$$\binom{2n}{n} \equiv \binom{2}{1}^{\omega_3(n)} \equiv (-1)^{\omega_3(n)} \pmod{3}. \qquad \blacksquare$$

Theorem 1 (D & S) *We have*

$$\binom{2n}{n} \equiv \left\{ \begin{array}{ll} (-1)^{\omega_3(n)} & \text{if } n \in T(01), \\ 0 & \text{else.} \end{array} \right\} \pmod 3$$

The *Thue-Morse sequence* is $\mathbf{t} = (t_0, t_1, t_2, \ldots)$ defined recursively by $t_0 = 0$ and for $n \geq 1$

$$(t_{2^n}, \ldots, t_{2^{n+1}-1}) = 1 - (t_0, \ldots, t_{2^n-1}).$$

So

$$\mathbf{t} = (0, 1, 1, 0, 1, 0, 0, 1, \ldots)$$

Benoit Cloitre conjectured the following.

Theorem 4 (D & S) *We have*

$$\left(\rho_3(n) \ : \ \binom{2n}{n} \equiv 1 \pmod 3\right) = \mathbf{t}$$

and

$$\left(\rho_3(n) \ : \ \binom{2n}{n} \equiv -1 \pmod 3\right) = 1 - \mathbf{t}.$$

Proof This follows easily by induction using the definition of $\mathbf{t}$ and Theorem 1. ∎

2. Catalan numbers

The *Catalan numbers* are

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

n	0	1	2	3	4	5	6	7	8
$\rho_2(C_n)$	1	1	0	1	0	0	0	1	0
$(n)_2$	$\emptyset$	1	10	11	100	101	110	111	1000

The *exponent* of n modulo 2 is

$$\xi_2(n) = \text{the largest power of 2 dividing } n.$$

Theorem 5 *We have*

$$\xi_2(C_n) = \omega_2(n+1) - 1.$$

Thus C_n is odd iff $n = 2^k - 1$ for some k.

Proof One can prove the displayed equation using Kummer's Theorem. D & S give a combinatorial proof using group actions on binary trees.

For the "Thus": C_n is odd iff $\xi_2(C_n) = 0$. But then $\omega_2(n+1) = 1$ which is iff $n+1 = 2^k$. ■

3. Motzkin numbers

The *Motzkin numbers* are

$$M_n = \sum_{k \geq 0} \binom{n}{2k} C_k.$$

A *run* in a sequence is a maximal subsequence of consecutive, equal elements.

Define a sequence $\mathbf{r} = (r_0, r_1, r_2, \ldots)$ by

$r_n =$ number of elements in the first n runs of $\mathbf{t}$.

Since

$$\mathbf{t} = (\hat{0}, \widehat{1, 1}, \hat{0}, \hat{1}, \widehat{0, 0}, 1, \ldots)$$

we have

$$\mathbf{r} = (1, 3, 4, 5, 7, \ldots).$$

The following theorem is implicit in a paper of Klazar & Luca.

Theorem 6 (D & S) *The Motzkin number M_n is even if and only if either $n \in 4\mathbf{r} - 1$ or $n \in 4\mathbf{r} - 2$.*

Proof Use induction and the description of M_n in terms of ordered trees where each vertex has at most 2 children. ■

4. Open problems

(i) Alter and Kubota have characterized the divisibility of C_n by primes and prime powers using Kummer's Theorem. Is it possible to find combinatorial proofs of their results for $p \geq 3$?

(ii) Explain the occurence of the sequence $\mathbf{r}$ in the characterization of the parity of M_n, e.g., by proving the result combinatorially.

(iii) D & S have characterized the residue of M_n modulo 3 and 5. What can be said for other primes or prime powers? The following conjecture is due in part to Amdeberhan.

Conjecture 7 (D & S) *The Motzkin number M_n is divisible by 4 if and only if*

$$n = (4i+1)4^{j+1} - 1 \text{ or } n = (4i+3)4^{j+1} - 2$$

for nonnegative integers i, j.

Furthermore we never have M_n divisible by 8.

n	0	1	2	3	4	5	6	7	8	9
$\rho_3\binom{2n}{n}$	1	2	0	2	1	0	0	0	0	2
$(n)_3$	$\emptyset$	1	2	10	11	12	20	21	22	100