# Chromatic symmetric functions and change of basis

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Let $G = (V, E)$ be a graph. Let $\mathbb{P}$ = positive integers, and $\mathbf{x} = \{x_1, x_2, \ldots\}$. For $c : V \to \mathbb{P}$ let

$$\mathbf{x}^c = \prod_{v \in V} x_{c(v)}.$$

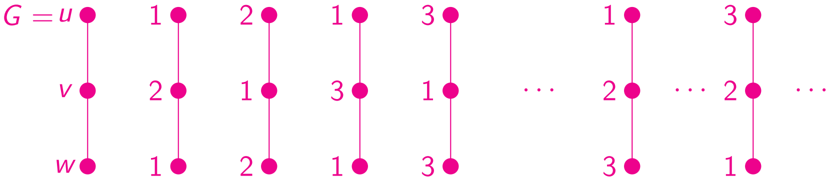
Call $c : V \to \mathbb{P}$ *proper* if

$$uv \in E \implies c(u) \neq c(v).$$

Stanley's *chromatic symmetric function* is

$$X(G) = X(G, \mathbf{x}) = \sum_{c\,:\,V \to \mathbb{P} \text{ proper}} \mathbf{x}^c.$$

**Ex.**

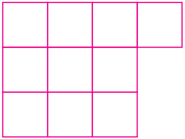


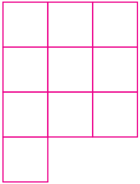

$$X(G) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 6x_1x_2x_3 + \cdots$$

A power series $f(\mathbf{x})$ is *symmetric* if it is invariant under permutation of variables.

**Ex.** $X(G, \mathbf{x})$ is symmetric for any graph $G$.

Bases for the algebra $\mathrm{Sym} = \mathrm{Sym}(\mathbf{x})$ of symmetric functions in $\mathbf{x}$ of bounded degree are indexed by partitions. A *partition* of a nonnegative integer $n$ is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\sum_i \lambda_i = n$ where the $\lambda_i$ are called *parts*. The *Young diagram* of $\lambda$ has $k$ left-justified rows with $\lambda_i$ boxes is row $i$. The *transpose* of $\lambda$ is $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots, \lambda_l^t)$ obtained by reflecting the Young diagram of $\lambda$ about the diagonal.

**Ex.** A partition of 10 is $\lambda = (4, 3, 3) =$ .

$\lambda^t =$  $= (3, 3, 3, 1)$.

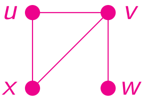
In graph $G = (V, E)$ a subset $I \subseteq V$ is *independent* if $u, v \in I$ implies $uv \notin E$. Note: if $c : V \to \mathbb{P}$ is proper then $c^{-1}(i)$ is independent for any $i \in \mathbb{P}$. The *independence number* of $G$ is

$$\alpha(G) = \text{maximum number of elements in an independent set of } G.$$

Subset $C \subseteq V$ is a *clique* if $u, v \in C$ implies $uv \in E$. The *clique number* of $G$ is

$$\omega(G) = \text{maximum number of elements in a clique of } G.$$

**Ex.** $G =$




$I = \{u, w\}$ is independent $\quad \alpha(G) = 2$

$C = \{u, v, x\}$ is a clique $\quad \omega(G) = 3$

The *monomial symmetric function basis* is defined by

$m_\lambda =$ sum of all monomials in $\mathbf{x}$ having exponent partition $\lambda$.

**Ex.** $m_{(2,1)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \cdots$.

Given any basis $\{b_\lambda\}$ for $\mathrm{Sym}(\mathbf{x})$ and $f(\mathbf{x}) \in \mathrm{Sym}(\mathbf{x})$ we say $b_\mu$ *appears* in $f(\mathbf{x})$ if $f(\mathbf{x}) = \sum_\lambda c_\lambda b_\lambda$ where $c_\mu \neq 0$.

Lemma (S-Tom)

*For a given $\lambda$, if $m_\lambda$ appears in $X(G)$ then*

*(a) $\alpha(G) \geq \lambda_1$, and*

*(b) $\omega(G) \leq \lambda_1^t$.*

Proof of (a). If $m_\lambda$ appears in $X(G)$ then the monomial $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ has nonzero coefficient. So there is a proper coloring $c : V \to \mathbb{P}$ with $\#c^{-1}(1) = \lambda_1$. Since $c^{-1}(1)$ is independent and $\alpha(G)$ is the maximum size of an independent set we have $\alpha(G) \geq \lambda_1$. $\square$

The *elementary symmetric function basis* is defined by

$$e_n = \text{sum of all square-free monomials in } \mathbf{x} \text{ of degree } n,$$
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}.$$

**Ex.** $$e_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots,$$
$$e_{(4,3,3)} = e_4e_3e_3.$$

Theorem ($\alpha/\omega$ Theorem, S-Tom)

*For a given $\lambda$, if $e_\lambda$ appears in $X(G)$ then*

*(a) $\omega(G) \leq \lambda_1$, and*

*(b) $\alpha(G) \geq \lambda_1^t$.*

Proof of (a). The basis change from $m_\mu$ to $e_\lambda$ has the form

$$m_\mu = \sum_{\lambda \geq \mu^t} c_{\lambda,\mu} e_\lambda.$$

where $\geq$ is lexicographic order. So if $e_\lambda$ appears in $X(G)$ then so does $m_\mu$ for some $\mu^t \leq \lambda$. Combined with the previous proposition part (b): $\omega(G) \leq \mu_1^t \leq \lambda_1$. □

Let $[m,n]=\{m,m+1,\ldots,n\}$ and $[n]=\{1,2,\ldots,n\}$. A *(natural unit) interval graph* $G$ has $V=[n]$ and for all $i<j$

$$ij\in E \implies [i,j] \text{ is a clique.}$$

**Ex.**


Given a basis $\{b_\lambda\}$ for $\mathrm{Sym}(\mathbf{x})$ say that $f(\mathbf{x})\in\mathrm{Sym}(\mathbf{x})$ is *b-positive* if each $b_\mu$ appearing in $f(\mathbf{x})$ has a positive coefficient. The next conjecture was made by Stanley and Stembridge for a wider class of graphs and Guay-Paquet refined it to interval graphs.

Conjecture (Stanley and Stembridge, Guay-Paquet)

*If $G$ is an interval graph then $X(G)$ is e-positive.*

Theorem (S-Tom)

*In any interval graph $G$ the coefficient of $e_\lambda$ in $X(G)$ is nonnegative if $\lambda_1\le 3$.*

Proof. If $e_\lambda$ appears in $X(G)$ then $\omega(G)\le 3$. Dahlberg has shown that such interval graphs are $e$-positive. □

Let $s_\lambda$ be the Schur basis for $\mathrm{Sym}$. If $G = ([n], E)$ is an interval graph then the coefficients of $X(G) = \sum_\lambda c_\lambda s_\lambda$ count $G$-tableaux. A *$G$-tableau of shape $\lambda$* is a bijective filling of the Young diagram of $\lambda$ with $[n]$ such that both the following hold.

1. If $ij$ is an adjacent pair in a row then $ij \notin E$ and $i < j$.
2. If $ij$ is an adjacent pair in a column and $ij \notin E$ then $i < j$.

**Ex.** $G =$


Some $G$-tableaux:

| | |
|---|---|
| 1 | 3 |
| 2 | 4 |

| | |
|---|---|
| 2 | 4 |
| 1 | |
| 3 | |

Some non-$G$-tableaux:

| | |
|---|---|
| 1 | 2 |
| 3 | 4 |

| | |
|---|---|
| 2 | 4 |
| 3 | |
| 1 | |

Let $T_\lambda(G)$ be the set of all $G$-tableau of shape $\lambda$.

Theorem (Gasharov)

*If $G$ is an interval graph and $X(G) = \sum_\lambda c_\lambda s_\lambda$ then*

$$c_\lambda = \# T_\lambda(G).$$

Theorem (dual Jacobi-Trudi determinant)

*If* $\lambda = (\lambda_1, \lambda_2, \ldots)$ *then* $s_{\lambda^t} = \begin{vmatrix} e_{\lambda_1} & e_{\lambda_1+1} & \cdots \\ e_{\lambda_2-1} & e_{\lambda_2} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}$.

So writing $X(G)$ first in $s_\lambda$ and then in $e_\mu$ has signed coefficients which count pairs $(T, \pi)$ where $T \in T_\lambda(G)$ and $\pi \in \mathfrak{S}_{\lambda_1}$ is the permutation from the determinant expansion.

**Ex.** Consider

$G =$ 


Then $\# T_\lambda(G) = 4$ for $\lambda = (2^2), (2, 1^2), (1^4)$.

$$\begin{aligned} X(G) &= 4s_{2^2} + 4s_{2,1^2} + 4s_{1^4} \\ &= 4\begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix} + 4\begin{vmatrix} e_3 & e_4 \\ e_0 & e_1 \end{vmatrix} + 4e_4 \\ &= 4e_{2^2} - 4e_{3,1} + 4e_{3,1} - 4e_4 + 4e_4 \\ &= 4e_{2^2}. \end{aligned}$$

We want a sign-reversing involution to cancel the negative terms. Let us concentrate on $e_n$.

Let $G = ([n], E)$ be an interval graph and $\lambda \vdash n$. The $e_h$ of largest subscript appearing in the determinant for $s_\lambda$ is at the end of the first row. And in that case $h$ is the hooklength of the $(1,1)$ box of the diagram of $\lambda$. So if $h = n$ then $\lambda$ is a hook. Furthermose $e_n$ only occurs with the permutation $\pi = c, 1, 2, \ldots, c-1$ where $c = \lambda_1$. So if $\lambda$ is a hook then let the *sign* of a $G$-tableau $T$ of shape $\lambda$ be

$$\operatorname{sgn} T = \operatorname{sgn} \lambda = (-1)^{c-1}.$$

If $\lambda$ is a hook then its *arm* and *leg* are the boxes in the first row, respectively first column, except (1,1).

**Ex.** $\lambda =$

| | | | | |
|---|---|---|---|---|
| | $A$ | $A$ | $A$ | $A$ |
| $L$ | | | | |
| $L$ | | | | |

$$s_\lambda = \begin{vmatrix} e_3 & e_4 & e_5 & e_6 & e_7 \\ e_0 & e_1 & e_2 & e_3 & e_4 \\ 0 & e_0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_0 & e_1 & e_2 \\ 0 & 0 & 0 & e_0 & e_1 \end{vmatrix}$$

$\pi = 51234$ $\qquad \operatorname{sgn} \lambda = (-1)^{5-1} = 1.$

$A =$ arm, $L =$ leg.

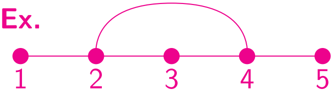
If $G = ([n], E)$ then an *inversion* in a $G$-tableau $T$ is a pair $ij$ with

1. $i < j$,
2. $i$ is in a lower row than $j$, and
3. $ij \in E$.

Let $\mathrm{Inv}\ T$ be the set of inversions of $T$. Call $k \in [n]$ *movable* in $T$ if it can be moved from the arm to the leg of $T$ or vice-versa with

1. the resulting tableau $T'$ being a $G$-tableau, and
2. $\mathrm{Inv}\ T = \mathrm{Inv}\ T'$.

**Ex.**




| | | |
|---|---|---|
| 1 | 3 | 5 |
| 2 | | |
| 4 | | |

$\mathrm{Inv}\ T = \{23, 45\}$.

3 is moveable and the resulting tableau is $T' =$

| | |
|---|---|
| 1 | 5 |
| 3 | |
| 2 | |
| 4 | |

.

Also 5 is moveable, but 2 and 4 are not.

## Lemma (S-Tom)

*If $k$ is moveable in $T$, then there is a unique position to which it can be moved.*

If $k$ is moveable in $T$ then let $T^k$ be the result of moving $k$. Define a map $\iota$ on $G$-tableau $T$ of hook shape by

$$\iota(T) = \begin{cases} T^k & \text{if } k \text{ is the smallest integer which is moveable in } T, \\ T & \text{if no element in } T \text{ is moveable.} \end{cases}$$

## Theorem (S-Tom)

*Let $G$ be an interval graph with $V = [n]$.*

1. *$\iota$ is a sign-reversing, $\mathrm{Inv}$-preserving, involution on hook $G$-tableaux.*
2. *If $T$ is fixed by $\iota$ then it has shape $1^n$.*
3. *The coefficient $c_n$ of $e_n$ in $X(G)$ is nonnegative. It is the number of $G$-tableaux of column shape with no moveable elements.*

**Other results obtained for $e_\lambda$ via the $m_\mu$.** We have a strengthening of the $\alpha/\omega$ Theorem using an analogue of Greene's invariant. One can also get results for other shapes.

Theorem (S-Tom)

*For any interval graph $G$ and $\lambda = (\mu, 1^k)$ where $\mu$ is any partition of $10$ or less and $k \geq 0$, the coefficient of $e_\lambda$ in $X(G)$ is nonnegative.*

**Other results for $e_\lambda$ via the Schur functions $s_\mu$.** It is known from the work of Stanley and the coefficient of $e_n$ in $X(G)$ for any graph $G$ is the number of acyclic orientations of $G$ with a single sink. We have a new interpretation of this coefficient for interval graphs in terms of $G$-tableaux. Others using this change of basis to make progress on the interval graph conjecture include Cho and Huh, Cho and Hong, and Wang.

# THANKS FOR LISTENING!