Bijective Proofs for Shuffle Compatibility

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Definitions

The method

Comments and open questions

Let $\mathbb{P} = \{1, 2, 3, \dots\}$ and let $S \subseteq \mathbb{P}$ be finite. A *permutation* of S is a linear ordering $\pi = \pi_1 \pi_2 \dots \pi_n$ of the elements of S. Let

$$L(S) = \{\pi \mid \pi \text{ a permutation of } S\}.$$

Ex. We have

$$L(\{2, 4, 7\}) = \{247,\ 274,\ 427,\ 472,\ 724,\ 742\}.$$

A *statistic* is a function st whose domain is $\uplus_S L(S)$. Examples include the *Descent set* of π which is

$$\mathrm{Des}\,\pi = \{i \ :\ \pi_i > \pi_{i+1}\}.$$

The *descent number* and *major index* of π are

$$\mathrm{des}\,\pi = |\,\mathrm{Des}\,\pi| \quad \text{and} \quad \mathrm{maj}\,\pi = \sum_{i \in \mathrm{Des}\,\pi} i$$

where $|\cdot|$ is cardinality.
Ex. If $\pi = 53698$ then

$$\mathrm{Des}\,\pi = \{1, 4\}, \quad \mathrm{des}\,\pi = 2, \quad \mathrm{maj}\,\pi = 1 + 4 = 5.$$

If π, σ are permutations with $\pi \cap \sigma = \emptyset$ then their *shuffle set* is

$$\pi \sqcup\!\sqcup \sigma = \{\tau \ : \ |\tau| = |\pi| + |\sigma| \text{ and } \pi, \sigma \text{ are subwords of } \tau\}.$$

Ex. We have

$$25 \sqcup\!\sqcup \mathbf{74} = \{25\mathbf{74},\ 2\mathbf{7}5\mathbf{4},\ 2\mathbf{74}5,\ \mathbf{7}25\mathbf{4},\ \mathbf{7}2\mathbf{4}5,\ \mathbf{74}25\}.$$

Statistic st is *shuffle compatible* if the multiset $\operatorname{st}(\pi \sqcup\!\sqcup \sigma)$ depends only on $|\pi|, |\sigma|, \operatorname{st}\pi$, and $\operatorname{st}\sigma$.

Ex. We have

$$\operatorname{des}(25 \sqcup\!\sqcup 74) = \{\{1, 1, 1, 1, 2, 2\}\} = \operatorname{des}(12 \sqcup\!\sqcup 43).$$

Shuffle compatibility is implicit in the work of Stanley on *P*-partitions. It was explicitly defined and studied using algebras whose multiplication involves shuffles by Gessel and Zhuang. Further work was done by Grinberg using enriched *P*-partitions, and by Oğuz who answered a question of Gessel and Zhuang. We have developed a method for proving shuffle compatibility bijectively.

A *descent statistic* is a statistic st such that $\mathrm{st}\,\pi$ only depends on $|\pi|$ and $\mathrm{Des}\,\pi$.

Ex. Statistics des and maj are descent statistics. The statistic inv is not: $\mathrm{Des}\,132 = \{2\} = \mathrm{Des}\,231$ but $\mathrm{inv}\,132 = 1$ and $\mathrm{inv}\,231 = 2$.

We use the notation

$$[n] = \{1, 2, \ldots, n\} \quad \text{and} \quad [n] + m = \{m+1, m+2, \ldots, m+n\}.$$

Lemma

Suppose st *is a descent statistic. The following are equivalent.*

(1) st *is shuffle compatible.*

(2) *If* $\mathrm{st}(\pi) = \mathrm{st}(\pi')$ *where* $\pi, \pi' \in L([m])$, $\sigma \in L([n] + m)$, *then*

$$\mathrm{st}(\pi \sqcup\!\sqcup \sigma) = \mathrm{st}(\pi' \sqcup\!\sqcup \sigma).$$

(3) *If* $\mathrm{st}(\sigma) = \mathrm{st}(\sigma')$ *where* $\sigma, \sigma' \in L([n] + m)$, $\pi \in L([m])$, *then*

$$\mathrm{st}(\pi \sqcup\!\sqcup \sigma') = \mathrm{st}(\pi \sqcup\!\sqcup \sigma).$$

Given π, π' with $|\pi| = |\pi'|$ and σ there is a *fundamental bijection*

$$\Phi : \pi \sqcup\!\sqcup\, \sigma \to \pi' \sqcup\!\sqcup\, \sigma$$

which replaces the elements of π in a shuffle by the elements of π'.

Ex. If $\pi = \mathbf{1423}$, $\pi' = \mathbf{2314}$ and $\sigma = 756$ then

$$\Phi(\mathbf{1}75\mathbf{4}6\mathbf{23}) = \mathbf{2}75\mathbf{3}1\mathbf{64}.$$

Theorem

Des is shuffle compatible.

Proof. By the lemma, it suffices to show that if $\pi, \pi' \in L([m])$, $\sigma \in L([n] + m)$ with $\operatorname{Des} \pi = \operatorname{Des} \pi'$ then

$$\operatorname{Des}(\pi \sqcup\!\sqcup\, \sigma) = \operatorname{Des}(\pi' \sqcup\!\sqcup\, \sigma).$$

This will be true if Φ preserves the descent set. We have $i \in \operatorname{Des} \tau$ for $\tau \in \pi \sqcup\!\sqcup\, \sigma$ iff $\tau_i \tau_{i+1}$ equals

1. $\pi_j \pi_{j+1}$ with $j \in \operatorname{Des} \pi$, or
2. $\sigma_k \sigma_{k+1}$ with $k \in \operatorname{Des} \sigma$, or
3. $\sigma_k \pi_j$.

It is easy to check that the same list holds for $\Phi(\tau)$. □

Permutation patterns. Given a permutation π let $\overline{\pi}$ be the corresponding consectuive pattern. Let $\overline{\Pi}$ be a set of consecutive patterns. Define a statistic on permutations σ by

$$\mathrm{st}_{\overline{\Pi}}(\sigma) = \text{number of copies of a } \overline{\pi} \in \overline{\Pi} \text{ in } \sigma.$$

Note that

$$\mathrm{st}_{\overline{21}}(\sigma) = \operatorname{des} \sigma$$

and

$$\mathrm{st}_{\{\overline{132}, \overline{231}\}}(\sigma) = \mathsf{pk}\, \sigma,$$

the number of peaks of σ. Both des and pk are shuffle compatible.

Question

For what $\overline{\Pi}$ is $\mathrm{st}_{\overline{\Pi}}$ shuffle compatible?

Multiple statistics. Gessel and Zhuang asked for criteria such that the shuffle compatibility of statistics st_1 and st_2 implies the shuffle compatibility of $(\mathrm{st}_1, \mathrm{st}_2)$. The map Φ also preserves Pk, Lpk, Rpk, and Epk (the set of peak sets of π, 0π, $\pi 0$, and $0\pi 0$, respectively). So Φ preserves any statistic formed from picking a tuple of these statistics.

Conjectured shuffle compatibility. Let $\mathsf{pk}\,\pi = |\,\mathsf{Pk}\,\pi|$ and

$$\mathsf{udr}\,\pi = \text{number of runs of } 0\pi,$$

where a run is a maximal increasing or decreasing factor. Gessel and Zhuang conjectured, and we proved, that $(\mathsf{pk}, \mathsf{udr})$ is shuffle compatible. See their paper for other conjectured shuffle compatible pairs and triples. In particular, is the triple $(\mathrm{des}, \mathsf{pk}, \mathsf{udr})$ shuffle compatible?

Repetitions. We have been assuming that $\pi \cap \sigma = \emptyset$. What happens if repetitions are allowed?

THANKS FOR

LISTENING!