Sign-reversing Involutions and the Chromatic Polynomial

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Sign-reversing involutions

Chromatic polynomials

Chromatic symmetric functions

Other applications

We let $[n] = \{1, 2, \ldots, n\}$ and S be a finite set. Bijection $\iota : S \to S$ is an *involution* if $\iota^2 = \mathrm{id}$. Any bijection $\iota : S \to S$ can be considered as a digraph with vertex set S and an arc $\vec{st}$ if $\iota(s) = t$. This graph can be decomposed into directed cycles.

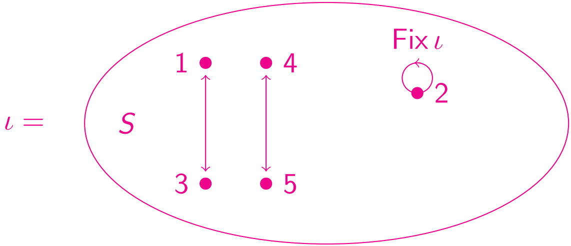
Lemma

ι is an involution iff every cycle contains 1 or 2 elements. □

Let

$$\operatorname{Fix} \iota = \{s \in S \mid \iota(s) = s\}.$$

Ex. Let $S = [5]$ and $\iota(1) = 3, \iota(2) = 2, \iota(3) = 1, \iota(4) = 5, \iota(5) = 4$. Then $\iota^2(1) = \iota(3) = 1$ and similarly $\iota^2(s) = s$ for all $s \in [5]$. The cycle containing 1 is $1 \leftrightarrow \iota(1)$ or $1 \leftrightarrow 3$. Also $\operatorname{Fix} \iota = \{2\}$.



A set S is *signed* if there is a map $\operatorname{sgn} : S \to \{-1, +1\}$. Let

$$S^+ = \{s \in S \mid \operatorname{sgn} s = +1\}, \qquad S^- = \{s \in S \mid \operatorname{sgn} s = -1\}.$$

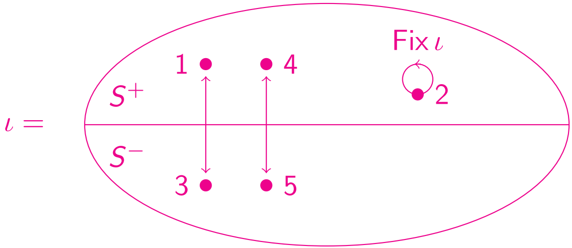
Involution $\iota : S \to S$ is *sign reversing* if

1. For every two-cycle $s \leftrightarrow t$ of ι we have $\operatorname{sgn} s = -\operatorname{sgn} t$.
2. For every fixed point s of ι we have $\operatorname{sgn} s = +1$.

So in this case, letting $\#$ denote cardinality,

$$\sum_{s \in S} \operatorname{sgn} s = \# \operatorname{Fix} \iota.$$

Ex. Let $\operatorname{sgn} 1 = \operatorname{sgn} 2 = \operatorname{sgn} 4 = +1$ and $\operatorname{sgn} 3 = \operatorname{sgn} 5 = -1$.



Theorem

For $n \geq 1$ we have

$$\sum_{k\geq 0}(-1)^k\binom{n}{k} = 0.$$

Proof. Let $S = 2^{[n]} = \{T \mid T \subseteq [n]\}$, and $\operatorname{sgn} T = (-1)^{\#T}$. So

$$\sum_{T\in S}\operatorname{sgn} T = \sum_{k\geq 0}\ \sum_{T\subseteq[n] \text{ and } \#T=k}(-1)^k = \sum_{k\geq 0}(-1)^k\binom{n}{k}.$$

Define a map $\iota : S \to S$ by symmetric difference

$$\iota(T) = T\Delta\{1\} = \begin{cases} T-\{1\} & \text{if } 1\in T,\\ T\cup\{1\} & \text{if } 1\notin T.\end{cases}$$

Then ι is an involution because $(T\Delta\{1\})\Delta\{1\} = T$.
Also $\operatorname{Fix}\iota = \emptyset$, and $\#(T\Delta\{1\}) = \#T \pm 1$ so $\operatorname{sgn}\iota(T) = -\operatorname{sgn} T$.
Thus ι is sign reversing and

$$\sum_{k\geq 0}(-1)^k\binom{n}{k} = \sum_{T\in S}\operatorname{sgn} T = \#\operatorname{Fix}\iota = 0. \qquad \square$$

Let $G = (V, E)$ be a finite graph. A *$[t]$-coloring* is a function $\kappa : V \to [t]$. The coloring is *proper* if $uv \in E$ implies $\kappa(u) \neq \kappa(v)$. The *chromatic polynomial* of G is

$$P(G; t) = \text{number of proper } \kappa : V \to [t].$$

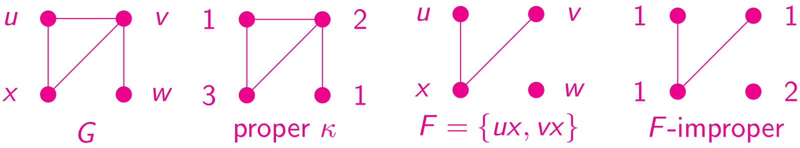
$F \subseteq E$ corresponds to the spanning subgraph of G with edges F. Coloring $\kappa : V \to [t]$ is *F-improper* if $uv \in F$ implies $\kappa(u) = \kappa(v)$.

Lemma

The number of F-improper colorings of G is $t^{c(F)}$ where $c(F)$ is the number of components of F.

Proof. Coloring κ is F-improper iff κ is constant on components of F. And each component has t color choices. □

Ex.



Theorem (Birkhoff)

For any graph $G = (V, E)$ we have

$$P(G;t) = \sum_{F \subseteq E} (-1)^{\#F} t^{c(F)}.$$

Proof. Let $S = \{(F, \kappa) \mid F \subseteq E \text{ and } \kappa : V \to [t] \text{ is } F\text{-improper}\}$.
If κ proper on G then $F = \emptyset$. Let $\operatorname{sgn}(F, \kappa) = (-1)^{\#F}$.

$$\sum_{(F,\kappa) \in S} \operatorname{sgn}(F, \kappa) = \sum_{F \subseteq E} (-1)^{\#F} \sum_{\kappa \text{ is } F\text{-improper}} 1 = \sum_{F \subseteq E} (-1)^{\#F} t^{c(F)}.$$

If κ is not proper then select $e_\kappa = uv \in E$ such that $\kappa(u) = \kappa(v)$.
Define a map $\iota : S \to S$ by symmetric difference

$$\iota(F, \kappa) = \begin{cases} (F, \kappa) & \text{if } \kappa \text{ proper,} \\ (F \Delta \{e_\kappa\}, \kappa) & \text{if } \kappa \text{ improper.} \end{cases}$$

Then ι is well defined since κ does not change and by the choice of e_κ. Also ι is a sign-reversing involution since Δ is and $\operatorname{sgn} \emptyset = +1$.

$$\therefore \sum_{(F,\kappa) \in S} \operatorname{sgn}(F, \kappa) = \# \operatorname{Fix} \iota = \#\{(\emptyset, \kappa) \mid \kappa \text{ proper}\} = P(G;t). \quad \square$$

Let $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$. A *symmetric function* is a power series $f(\mathbf{x})$ of bounded degree invariant under permuting the variables.

Ex. $f(\mathbf{x}) = 3x_1x_2^2 + 3x_1^2x_2 + \cdots + 5x_1x_2x_3 + 5x_1x_2x_4 + \cdots$.

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers.

Ex. $\lambda = (5, 5, 3, 2, 2, 1)$.

Bases for the algebra of symmetric functions are indexed by partitions. The *power sum basis*, p_λ, is defined by

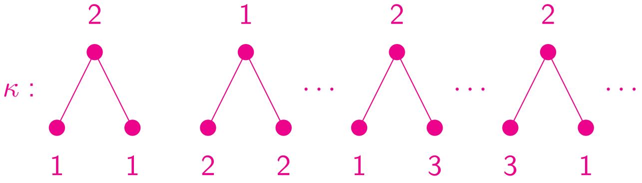
1. $p_n = x_1^n + x_2^n + \cdots$, and
2. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ then $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$.

Ex. $p_{(5,5,3)} = p_5p_5p_3 = (x_1^5 + x_2^5 + \cdots)^2(x_1^3 + x_2^3 + \cdots)$.

The *chromatic symmetric function* of $G = (V, E)$ is

$$X(G; \mathbf{x}) = \sum_{\text{proper } \kappa : V \to \{1,2,3,\ldots\}} \prod_{v \in V} x_{\kappa(v)}.$$

Ex. For G being a path of length 2:



$$X(G; \mathbf{x}) = x_1^2 x_2 + x_1 x_2^2 + \cdots + 6 x_1 x_2 x_3 + \cdots$$

Note that setting $x_1 = \ldots = x_t = 1$ and $x_i = 0$ for $i > t$ gives $X(G; \mathbf{x}) = P(G; t)$. Associate with $F \subseteq E$ the partition $\lambda(F)$ where the λ_i are the number of vertices in the components of F.

Theorem (Stanley)

For any graph $G = (V, E)$ we have

$$X(G; \mathbf{x}) = \sum_{F \subseteq E} (-1)^{\#F} p_{\lambda(F)}. \qquad \square$$

(a) There is a theorem of Whitney characterizing the coefficients of $P(G;t)$ in terms of NBC (no broken circuit) sets. Stanley generalized this result to $X(G;\mathbf{x})$. We give proofs of both using sign-reversing involutions.

(b) An *orientation* $\mathcal{O}$ of G is a digraph obtained by replacing each edge of G by an arc. Say $\mathcal{O}$ is *acyclic* if it contains no directed cycles. Using the NBC theorem for $P(G;t)$, Blass and S gave a bijective proof of the the following theorem.

Theorem (Stanley)

For any graph G with $\#V = n$ we have

$$P(G;-1) = (-1)^n(\textit{number of acyclic orientations of } G). \qquad \square$$

Stanley proved generalizations of this result to $P(G;-t)$ for any t and to $X(G;\mathbf{x})$. S and Vatter proved these bijectively.

(c) There is a generalization of Whitney's NBC theorem to geometric lattices. Blass and S generalized this even further to arbitrary lattices.

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THANKS FOR LISTENING!