

Stirling numbers for complex reflection groups

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Ordinary Stirling numbers

Complex reflection groups

Partitions again

Future directions

Let $[n] = \{1, 2, \dots, n\}$. A *partition of $[n]$ into k blocks* is $\rho = S_1 / \dots / S_k$ where $[n] = \uplus_i S_i$ and $S_i \neq \emptyset$ for all i . The *Stirling numbers of the second kind* are

$$S(n, k) = \#\{\rho \mid \rho \text{ is a partition of } [n] \text{ into } k \text{ blocks}\}.$$

Ex. If $n = 3$ then

k	1	2	3
ρ	123	1/23, 2/13, 3/12	1/2/3
$S(3, k)$	1	3	1

Let \mathfrak{S}_n denote the symmetric group of permutations π of $[n]$. The *Stirling numbers of the first kind* are

$$s(n, k) = (-1)^{n-k} \#\{\pi \mid \pi \in \mathfrak{S}_n \text{ has } k \text{ disjoint cycles}\}.$$

Ex. If $n = 3$ then

k	1	2	3
π	(1, 2, 3), (1, 3, 2)	(1)(2, 3), (2)(1, 3), (3)(1, 2)	(1)(2)(3)
$s(3, k)$	2	-3	1

Let $\mathbf{x}_n = \{x_1, \dots, x_n\}$ be a set of commuting variables. The *degree* of a monomial $m = x_1^{k_1} \dots x_n^{k_n}$ is $\deg m = \sum_i k_i$. Define *complete homogeneous symmetric functions* by

$$h_k(\mathbf{x}_n) = \sum_{\deg m=k} m.$$

Ex.

k	1	2	3
$h_{3-k}(\mathbf{x}_k)$	$h_2(\mathbf{x}_1) = x_1^2$	$h_1(\mathbf{x}_2) = x_1 + x_2$	$h_0(\mathbf{x}_3) = 1$
$h_{3-k}(1, \dots, k)$	1	3	1

Proposition

We have $S(n, k) = h_{n-k}(1, 2, \dots, k)$.

Proof. Induct on n using the recursions

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

and

$$h_k(\mathbf{x}_n) = h_k(\mathbf{x}_{n-1}) + x_n h_{k-1}(\mathbf{x}_n)$$

to get the result. □

Define *elementary symmetric functions* by

$$e_k(\mathbf{x}_n) = \sum_{\deg m=k, m \text{ square free}} m.$$

Ex.

k	1	2	3
$(-1)^{3-k} e_{3-k}(\mathbf{x}_2)$	$e_2(\mathbf{x}_2) = x_1 x_2$	$-e_1(\mathbf{x}_2) = -(x_1 + x_2)$	$e_0(\mathbf{x}_2) = 1$
$(-1)^{3-k} e_{3-k}(1, 2)$	2	-3	1

Proposition

We have $s(n, k) = (-1)^{n-k} e_{n-k}(1, 2, \dots, n-1)$.



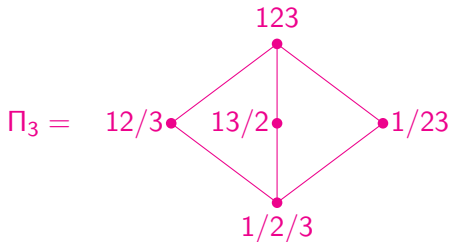
Let P be a finite poset with a minimum element $\hat{0}$, and a *rank function* where for $x \in P$

$$\text{rk } x = \text{length of any maximal } \hat{0}\text{-}x \text{ chain.}$$

Let

$\Pi_n =$ set of partitions ρ of $[n]$ ordered by refinement.

Ex. if $n = 3$ then



So if $\rho = S_1 / \dots / S_k \in \Pi_n$ then

$$\text{rk } \rho = n - k.$$

The *Whitney numbers of the 2nd kind for P* are

$$W(P, k) = \sum_{\text{rk } x = k} 1 = \#\{x \in P \mid \text{rk } x = k\}.$$

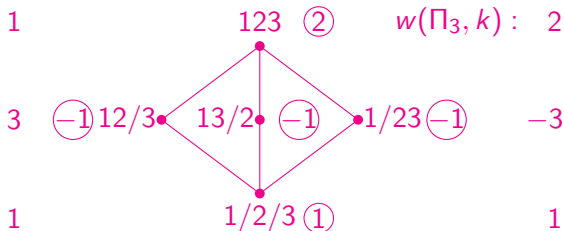
The *Möbius function of P* is defined by $\mu(\hat{0}) = 1$ and for $x > \hat{0}$

$$\mu(x) = -\sum_{y < x} \mu(y) \iff \sum_{y \leq x} \mu(y) = \delta_{x, \hat{0}}.$$

The *Whitney numbers of the 1st kind for P* are

$$w(P, k) = \sum_{\text{rk } x = k} \mu(x).$$

Ex. $W(\Pi_3, k) :$ 1



Proposition

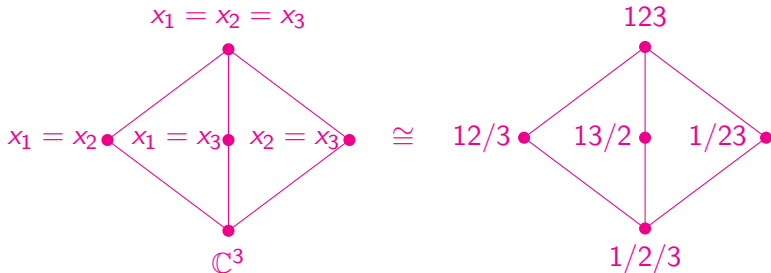
We have $W(\Pi_n, k) = S(n, n - k)$ and $w(\Pi_n, k) = s(n, n - k)$. □

A *hyperplane* in \mathbb{C}^n is a subspace H with $\dim H = n - 1$. A *hyperplane arrangement* is a finite set $\mathcal{A} = \{H_1, \dots, H_k\}$ of hyperplanes. The *braid arrangement* in \mathbb{C}^n is

$$Br_n = \{x_i = x_j \mid 1 \leq i < j \leq n\}.$$

The *intersection lattice* $L(\mathcal{A})$ of an arrangement is all subspaces $W \subseteq \mathbb{C}^n$ which can be obtained as the intersection of some of the hyperplanes in \mathcal{A} ordered by reverse inclusion.

Ex. We have $Br_3 = \{x_1 = x_2, x_1 = x_3, x_2 = x_3\}$, with lattice



Proposition

We have $L(Br_n) \cong \Pi_n$ as posets.



A *pseudoreflexion* is a linear map $M : \mathbb{C}^n \rightarrow \mathbb{C}^n$ which fixes a hyperplane and is of finite order. A *complex reflection group* G is a group generated by pseudoreflections. Call G *irreducible* if its only G -invariant subspaces are \mathbb{C}^n and the origin, and n is called G 's *rank*. Shephard and Todd classified the finite irreducible complex reflection groups into 3 infinite families and 34 exceptionals.

$G(m, p, n) :=$ group of all $n \times n$ complex matrices M satisfying

1. Each row and column of M contains exactly one nonzero entry, say ζ_i in row i .
2. Each ζ_i is an m th root of unity.
3. We have $p|m$ and $(\zeta_1 \cdots \zeta_n)^{m/p} = 1$.

Ex. If

$$M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ ia \\ b \end{bmatrix} = \begin{bmatrix} a \\ ia \\ b \end{bmatrix}.$$

So M fixes $x_2 = ix_1$ and $M^2 = I$. Also $M \in G(4, p, 3)$ for any $p|4$.

Note that in $G(1, 1, n)$ we have $\zeta_i = 1$. So $G(1, 1, n) \cong \mathfrak{S}_n$. This is called *type A*.

Given any finite complex reflection group G we let

$$\mathcal{A}(G) = \{H \mid H \text{ a fixed hyperplane of a pseudoreflection in } G\},$$

$$L(G) = \text{intersection lattice of } \mathcal{A}(G).$$

If G is irreducible of rank n then it's *Stirling numbers of the first and second kinds* are, respectively,

$$s(G, k) = w(L(G), n - k) \quad \text{and} \quad S(G, k) = W(L(G), n - k).$$

Theorem (Orlik-Solomon, 1980)

If G is a finite, irreducible complex reflection group with coexponents e_1^, \dots, e_n^* then*

$$s(G, k) = (-1)^{n-k} e_{n-k}(e_1^*, \dots, e_n^*).$$

For $S(G, k)$ things are more complicated.

Lemma

The reflecting hyperplanes of $G(m, p, n)$ are of the form

1. $x_i = \zeta x_j$ for $\zeta^m = 1$ and distinct $i, j \in [n]$,
2. $x_i = 0$ for $i \in [n]$ in the case $p < m$.

Ex. In $G(4, 1, 3)$ the pseudoreflections

$$M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}$$

have corresponding hyperplanes

$$x_2 = ix_1 \quad \text{and} \quad x_3 = 0.$$

Theorem (S-Swanson)

Let $G = G(m, p, n)$.

$$S(G, k) = \begin{cases} h_{n-k}(1, m+1, \dots, km+1) := h(m, k, n) & \text{for } p < m, \\ h(m, k, n) - nh_{n-k-1}(m, 2m, \dots, km) & \text{for } p = m. \end{cases}$$

Is there a way to interpret $S(G, k)$ in terms of partitions? Consider $G = G(2, 1, n) = B_n$. The hyperplanes of B_n are of three types

$$x_i = x_j, \quad x_i = -x_j, \quad x_i = 0.$$

Corresponding partitions ρ of $\langle n \rangle = \{0, \pm 1, \dots, \pm n\}$ will have

1. a block containing i, j and a different block containing $-i, -j$,
2. a block containing $i, -j$ and a different block containing $-i, j$,
3. the block containing 0 also contains $\pm i$.

Ex. In \mathbb{C}^5 subspace $(x_1 = x_3 = -x_4) \cap (x_5 = 0)$ has partition

$$\rho = 0, -5, 5 / 1, 3, -4 / -1, -3, 4 / 2 / -2.$$

Partition $\rho = S_0/S_1/S_2/\dots/S_{2k}$ of $\langle n \rangle$ is *type B_n* if

1. $0 \in S_0$, and if $i \in S_0$ then also $-i \in S_0$,
2. $S_{2m} = -S_{2m-1}$ for $m \geq 1$.

Theorem (Zaslavsky, 1982)

$S(B_n, k)$ is the number of type B_n partitions with $2k + 1$ blocks.

S and Swanson have a generalization of this result to all $G(m, p, n)$ partitions of the elements of $[n]$ colored in m colors.

1. **q -Stirlings.** We have defined q -analogues of $s(G, k)$ and $S(G, k)$ for $G = G(m, p, n)$ using an inversion-like statistic. These polynomials satisfy various identities and log-concavity conjectures.
2. **A new symmetric function identity.**

Theorem (S-Swanson)

$$t^n = \sum_{k=0}^n h_{n-k}(\mathbf{x}_{k+1}) \cdot (t - x_1)(t - x_2) \cdots (t - x_k).$$

3. **If not h ...** For some complex reflection groups (e.g., those corresponding to E_6, E_7, E_8), we have been unable to find a nice expression for $S(G, k)$ in terms of the $h_k(\mathbf{x}_n)$.
4. **Generating functions.** We derive ordinary, exponential, and q -exponential generating functions for the Stirling numbers, some of which are new even in type A. Proof techniques include species, q -difference equations, and sign-reversing involutions.
5. **Coinvariant algebras.** The $S(G, k)$ for $G(m, 1, n)$ appear in conjectured Hilbert series for corresponding super coinvariant algebras. We give conjectured analogues of the Artin basis for these algebras indexed by colored partitions.

THANKS FOR
LISTENING!