Permutation Patterns and Statistics

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joint work with

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Outline

Two sequences of distinct integers $\pi = a_1 a_2 \ldots a_k$ and $\sigma = b_1 b_2 \ldots b_k$ are *order isomorphic* if, for all i and j,

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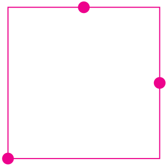
$132 =$ 

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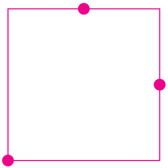
$132 =$ 

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$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_0, r_1, r_{-1}, r_\infty\}$$

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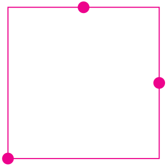

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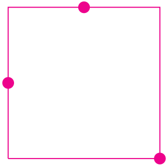

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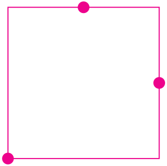
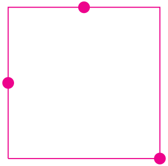
$132 =$ 

$R_{90}(132) =$  $= 231$

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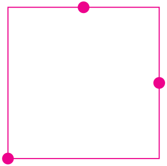
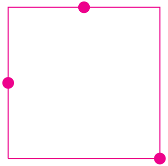
$132 =$  $R_{90}(132) =$  $= 231$

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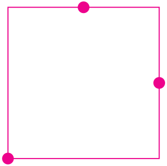
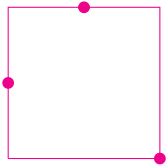
$132 =$  $R_{90}(132) =$  $= 231$

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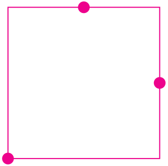
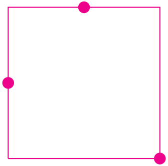
$132 =$ 

$R_{90}(132) =$  $= 231$

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Proposition

For any $\rho \in D_4$ and any permutation π we have

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Outline

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Note: No $\rho \in D_4$ preserves the major index.

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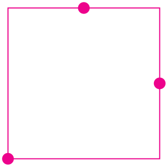
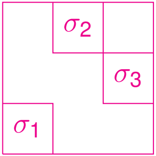
$132 =$ 

$132[\sigma_1, \sigma_2, \sigma_3] =$ 

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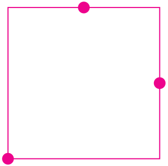
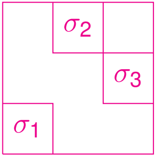
$132 =$ 

$132[\sigma_1, \sigma_2, \sigma_3] =$ 

Conjecture

For all $m, n \geq 0$ we have: $132[\iota_m, 1, \delta_n] \overset{\text{maj}}{\equiv} 231[\iota_m, 1, \delta_n]$, *where $\iota_m = 12\ldots m$ and $\delta_n = n(n-1)\ldots 1$.*

Outline

Since $\# \text{Av}_n(\pi) = C_n$ for any $\pi \in \mathfrak{S}_3$, the corresponding $I_n(\pi; q)$ and $M_n(\pi; q)$ are q-analogues of the Catalan numbers since setting $q = 1$ we recover C_n.

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$$\begin{aligned} C_n(q) &= I_n(132; q) = I_n(213; q), \\ \tilde{C}_n(q) &= I_n(231; q) = I_n(312; q) \end{aligned}$$

were introduced by Carlitz and Riordan and studied by numerous authors but the others seem to be new.

Since $\# \operatorname{Av}_n(\pi) = C_n$ for any $\pi \in \mathfrak{S}_3$, the corresponding $I_n(\pi; q)$ and $M_n(\pi; q)$ are q-analogues of the Catalan numbers since setting $q = 1$ we recover C_n. The polynomials

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Theorem (DDJSS)

For $n \geq 1$ we have

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Theorem (DDJSS)

For all $k \geq 0$ we have

$$\langle q^i \rangle I_{2^k-1}(321; q) = \begin{cases} 1 & \text{if } i = 0, \\ \text{an even number} & \text{if } i \geq 1. \end{cases}$$

Outline

1. (a) Use symmetries of the square to prove $123 \equiv 321$ and $132 \equiv 213 \equiv 231 \equiv 312$.

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(e) Let $P(n, k)$ denote the set of w permutations of k zeros and $n-k$ ones. Defining inv similarly to $\mathfrak{S}_n$, prove

$$\sum_{w \in P(n,k)} q^{\operatorname{inv} w} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

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