Patterns and Statistics

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu

www.math.msu.edu/~sagan

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Outline

The philosophy.

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By combining the the theory of patterns with the theory of statistics, one opens up a whole realm of research problems waiting to be explored.

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4. st = lb, the left-bigger statistic.

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3. Study properties of the $\mathrm{ST}_n(t;q)$ which have no analogues when $q = 1$ such as degree, coefficients, unimodality, log concavity, real rootedness and so forth.

Papers with work in this area	S_n	st
Bach, Remmel	$\mathfrak{S}_n$	des, lrm
Barnabei, Bonetti, Elizalde, Silimbani	$\mathfrak{S}_n$	maj
Baxter	$\mathfrak{S}_n$	maj, peak, valley
Bloom	$\mathfrak{S}_n$	maj
Bousquet-Mélou	P_n	level, min, minmax
Chan/Trongsiriwat	$\mathfrak{S}_n$	inv
Chen, Dai, Dokos, Dwyer, S	Asc_n	asc, rlm
Chen, Elizalde, Kasraoui, S	$\mathfrak{S}_n$	inv, maj
Dahlberg, S	$\mathcal{I}_n$	inv, maj
Dahlberg, Dorward, Gerhard, Grubb, Purcell, Reppuhn, S	Π_n	ls, lb, rs, rb
Dokos, Dwyer, Johnson, S, Selsor	$\mathfrak{S}_n$	inv, maj
Duncan, Steingrímsson	Asc_n	asc, rlm
Elizalde (also with Deutsch, Pak)	$\mathfrak{S}_n$	des, exc, fp,
Goyt (with Mathisen, S)	Π_n	ls, rb
Killpatrick	$\mathfrak{S}_n$	ch, maj
Kitaev, Remmel	P_n	level, min
Stanton, Simion	Π_n	ls, lb, rs, rb

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Let $[n] = \{1, 2, \ldots, n\}$,

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Permutation $\sigma = a_1 a_2 \ldots a_n$ has *descent set/descent number*

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$$\operatorname{Des} \sigma = \{2, 5, 7\}, \qquad \operatorname{des} \sigma = 3, \qquad \operatorname{maj} \sigma = 2 + 5 + 7 = 14.$$

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$$\begin{array}{cccccccccccc} \sigma & : & 123 & & 132 & & 213 & & 231 & & 312 \\ \operatorname{maj} \sigma & : & 0 & & 2 & & 1 & & 2 & & 1 \\ M_3(321) & = & q^0 & + & q^2 & + & q^1 & + & q^2 & + & q^1. \end{array}$$

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Dokos, Dwyer, Johnson, Selsor, and S (DDJSS) where the first authors to comprehensively study $M_n(\pi)$ for all $\pi \in \mathfrak{S}_3$ as well as similarly defined polynomials for multiple pattern avoidance and for the inversion statistic.

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Theorem (DDJSS)

The maj-Wilf equivalence classes for $\pi \in \mathfrak{S}_3$ are

$$\begin{aligned} [123]_{\text{maj}} &= \{123\}, \\ [321]_{\text{maj}} &= \{321\}, \\ [132]_{\text{maj}} &= \{132, 231\}, \\ [213]_{\text{maj}} &= \{213, 312\}. \end{aligned}$$

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The maj-Wilf equivalence classes for $\pi \in \mathfrak{S}_3$ are

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Proof. To show there are no other maj-Wilf equivalences, compare the polynomials $M_3(\pi)$ for $\pi \in \mathfrak{S}_3$. For $\sigma = a_1 \dots a_n$ let $\sigma^c = (n+1-a_1)\dots(n+1-a_n)$. So $\operatorname{Des}\sigma^c = [n-1] - \operatorname{Des}\sigma$

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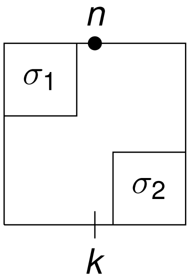
$\sigma =$ 

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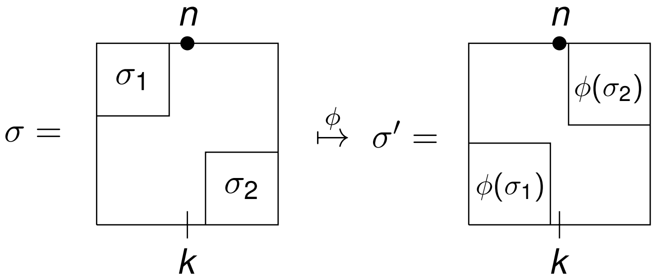

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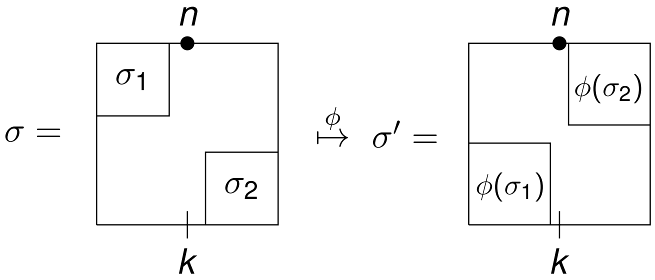

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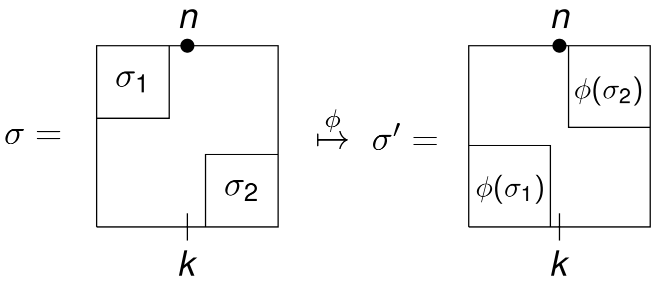

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This map has also been used by Bouvel and Viennot.

If $\pi = a_1 \dots a_n$ and $\sigma_1, \dots, \sigma_n \in \mathfrak{S}$ then the *inflation* of π by the σ_i is the permutation $\pi[\sigma_1, \dots, \sigma_n]$ whose diagram is obtained from that of π by replacing the ith dot with a copy of σ_i for all i.

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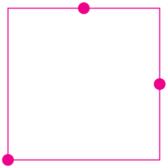

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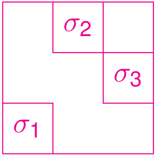

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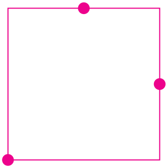
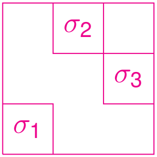
$132 =$ 

$132[\sigma_1, \sigma_2, \sigma_3] =$ 

Conjecture (DDJSS)

For all $m, n \geq 0$ we have:

$$132[\iota_m, 1, \delta_n] \equiv_{\text{maj}} 231[\iota_m, 1, \delta_n],$$

where $\iota_m = 12 \dots m$ and $\delta_n = n(n-1) \dots 1$.

Outline

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Killpatrick's proof uses the charge statistic of Lascoux and Schützenberger.

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Theorem (DDJSS)

$$M(231, 321; q, x) = \sum_{k \geq 0} \frac{q^{k^2} x^{2k}}{(x)_k (x)_{k+1}}.$$

IV. References

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Pick your favorite pattern avoidance notion and favorite statistic and have them play together!

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THANKS FOR LISTENING!