Pattern-Avoiding Polytopes and Bruhat Orders II

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Last time. . .

- $B_n(132, 312)$ is the polytope whose vertices are permutations matrices corresponding to elements of $\mathrm{Av}_n(132, 312)$
- $Q_n(132, 312)$ is the poset on $\mathrm{Av}_n(132, 312)$ with the right weak Bruhat order
  - This is isomorphic to the distributive lattice $M(n-1)$ of shifted Young tableaux contained inside $(n-1, n-2, \ldots, 1)$
- The simplicial complex $\mathcal{T}_n(132, 312)$, induced from $\Delta(Q_n(132, 312))$, is shellable and consists of unimodular simplices with vertices from $B_n(132, 312)$

This time. . .

1. Show that $\mathcal{T}_n(132, 312)$ is geometrically a triangulation of $B_n(132, 312)$
2. Create an EL-labeling of $Q_n(132, 312)$ to help describe the $h^*(B_n(132, 312))$
3. Use the theory of (Q, ω)-partitions to show that the h^*-vector is symmetric
4. Draw additional conclusions about the h^*-vector and the normalized volume of the polytope

Gorenstein polytopes

Theorem (D. and Sagan)

$\mathcal{T}_n(132, 312)$ is a unimodular, shellable, regular, reverse lexicographic triangulation of $B_n(132, 312)$.

Theorem (Stanley, 1978)

If P is a lattice polytope, then $h^*(P)$ is symmetric if and only if P is Gorenstein.

Sometimes it's easy to check if P is Gorenstein – but not this time. So we'll obtain symmetry in another way.

Definition

Let Δ be a d-dimensional abstract simplicial complex, and let f_i denote the number of i-dimensional faces of Δ. The h-vector of Δ is the sequence $h(\Delta) = (h_0, \ldots, h_d)$ defined by

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}.$$

Note: if $\mathcal{T}$ is a triangulation of a polytope, then $\mathcal{T}$ also has a simplicial complex structure, so writing $h(\mathcal{T})$ makes sense.

Theorem (Stanley, 1978)

If $\mathcal{T}$ is a geometric, unimodular, reverse lexicographic triangulation of P, then $h^*(P) = h(\mathcal{T})$.

Finding shelling numbers

Definition

Suppose $T_1, \dots, T_k$ is a shelling order of the maximal simplices in a triangulation of a polytope. The shelling number of T_j is

$$r(T_j) = \#\{v \in T_j \mid (T_j \setminus v) \subseteq (T_1 \cup \cdots \cup T_{j-1})\}.$$

In other words, $r(T_j)$ is the number of facets of T_j that glue into $T_1 \cup \cdots \cup T_{j-1}$.

Theorem (Stanley, 1978)

Suppose $T_1, \dots, T_k$ is a shelling order of a simplicial complex Δ. Then the component h_i of $h(\Delta)$ is the number of T_j such that $r(T_j) = i$.

Finding shelling numbers

Lemma (Björner, 1980)

If c is a maximal chain in a poset Q admitting an EL-labeling λ, then

$$r(\Delta(c)) = \operatorname{des} \lambda(c)$$

where des is number of descents.

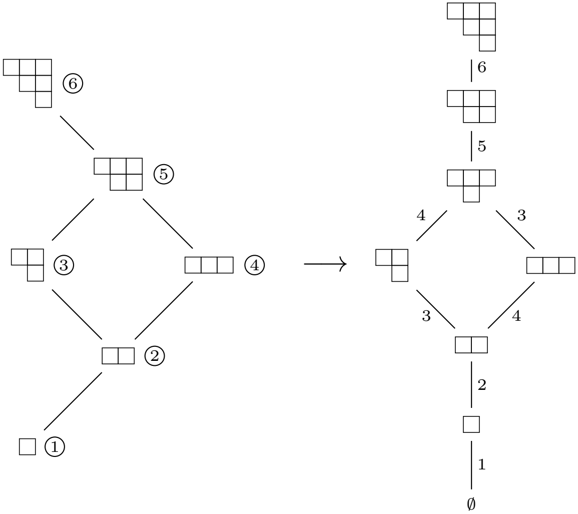
Goal is now to find a specific EL-labeling of $Q_n(132, 312)$.

Theorem (Stanley, 1972)

Suppose Q is a distributive lattice, $\operatorname{Irr}(Q)$ is its poset of join-irreducibles, and $\#(\operatorname{Irr}(Q)) = k$. If $f : \operatorname{Irr}(Q) \to [k]$ is an order-preserving map, then f induces an EL-labeling of Q.

General idea: a covering $I \lessdot J$ of order ideals in $\operatorname{Irr}(Q)$ is labeled with $J \setminus I$.

Example: EL-labeling of $M(3)$



(Q, ω)-partitions

Let Q be a poset on n elements and $\omega : Q \to [n]$ a bijection. A dual (Q, ω)-partition is a function $f : Q \to [m]$ such that

1. f is order-preserving, and
2. if $s < t$ and $\omega(s) > \omega(t)$, then $f(x) < f(t)$.

The order polynomial $\Omega_{Q,\omega}(m)$ is the number of functions f satisfying the above conditions.

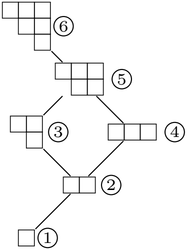
(Q,ω)-partitions

Definition

Given a poset Q with n elements and a labeling ω, its Jordan-Hölder set, $\mathcal{L}(Q,\omega)$, is the set of permutations

$$w = \omega(q_1)\omega(q_2)\dots\omega(q_n)$$

where $q_1,\dots,q_n$ runs over all linear extensions of Q.



$$\mathcal{L}(M(3),\omega) = \{123456, 124356\}$$

(Q,ω)-partitions

Theorem (Stanley (EC1))

Let Q be a poset and ω a natural labeling of Q (i.e. an order-preserving bijection).

1. We may write

$$\sum_{m\geq 0}\Omega_{Q,\omega}(m)t^m = \frac{\sum_{w\in\mathcal{L}(Q,\omega)} t^{1+\operatorname{des} w}}{(1-t)^{\#Q+1}}$$

2. The coefficients in the numerator above are symmetric if and only if Q is graded.

Theorem (D. and Sagan)

For all n, $h^*(B_n(132, 312))$ is symmetric.

Proof.

Earlier said that h_i^* counts the number of sequences of edge labels in maximal chains of $Q_n(132, 312)$ with i descents. These sequences are exactly the linear extensions of $\mathrm{Irr}(Q_n(132, 312))$, so

$$\sum_{i=0}^{d} h_i^* t^i = \frac{1}{t}\left(\sum_{w \in \mathcal{L}(Q,\omega)} t^{1+\operatorname{des} w}\right)$$

Since $\mathrm{Irr}(Q_n(132, 312))$ is graded, the coefficients on each side are symmetric. □

Our Rewards!

Corollary (D. and Sagan)

For all n, $B_n(132, 312)$ is Gorenstein.

Theorem (Bruns and Römer, 2007)

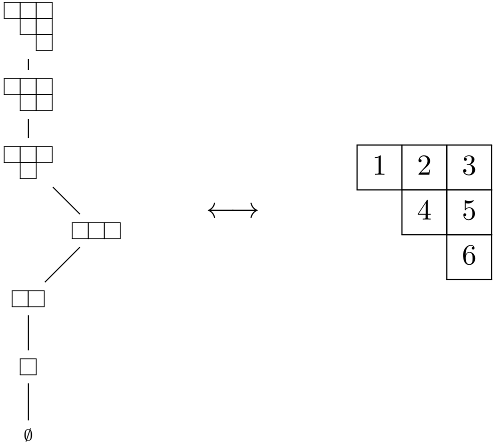
Every Gorenstein lattice polytope with a regular unimodular triangulation has a unimodal h^*-vector.

Corollary (D. and Sagan)

For all n, $h^*(B_n(132, 312))$ is unimodal.

Our Rewards!

Maximal chains in $Q_n(132, 312)$ are in bijection with shifted standard Young tableaux:



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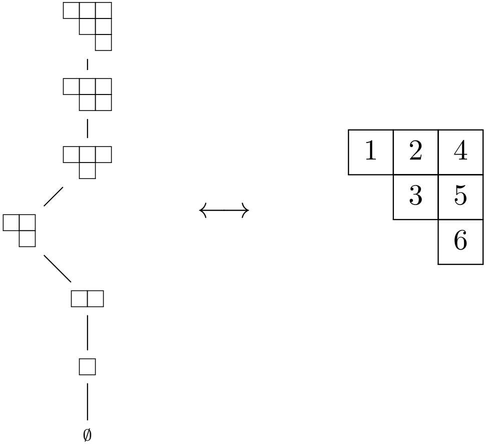

Our Rewards!

Fact: the normalized volume of a lattice polytope is $\sum h_i^*$.
So, by the hook length formula for shifted standard Young tableaux...

Corollary (D. and Sagan)

The normalized volume of $B_n(132, 312)$ is

$$\operatorname{Vol} B_n(132, 312) = \binom{n}{2}! \frac{\prod_{i=1}^{n-1}(i-1)!}{\prod_{i=1}^{n-1}(2i-1)!}$$

Some Wide-Open Questions

1. For "nice" special classes of Π,
    1. what is the combinatorial structure of $B_n(\Pi)$?
    2. what is $\mathrm{Vol}(B_n(\Pi))$?
    3. what is the h^*-vector of $B_n(\Pi)$?
2. What happens if we consider vincular or bivincular patterns? Other kinds of patterns?
3. For which choices of Π is $B_n(\Pi)$ Gorenstein?
4. What are the homotopy types of $Q_n(\Pi)$? (in general their order complexes aren't necessarily spheres, or even Cohen-Macaulay)